① $\dfrac{1}{3x^2-2x-1} = \dfrac{A}{x-1} + \dfrac{B}{3x+1} \quad \Big/ \cdot 3x^2-2x-1$

$$1 = A(3x+1) + B(x-1)$$

$$\Rightarrow \quad A = \frac{1}{4}, \; B = -\frac{3}{4}$$

$$I = \frac{1}{4}\int \frac{dx}{x-1} - \frac{3dx}{3x+1} = \frac{1}{4}\ln\left|\frac{x-1}{3x+1}\right| \quad \text{m.x.}$$

$x \in (-\infty, -\frac{1}{3})$
$(-\frac{1}{3}, 1)$
$(1, +\infty)$.

② $\dfrac{1}{2}\displaystyle\int \frac{2x}{x^4-2x^2-1} = \frac{1}{2}\int \frac{dy}{y^2-2y-1}$;

$$y^2-2y-1 = (y-1)^2-2 = \left(y-(1+\sqrt{2})\right)\left(y-(1-\sqrt{2})\right)$$

$$\frac{1}{y^2-2y-1} = \frac{A}{y-(1+\sqrt{2})} + \frac{B}{y-(1-\sqrt{2})} \;; \quad \boxed{A = -B = \frac{1}{2\sqrt{2}}}$$

$$I = \frac{1}{4\sqrt{2}}\int \frac{1}{y-(1+\sqrt{2})} - \frac{1}{y-(1-\sqrt{2})}\,dy = \frac{1}{4\sqrt{2}}\ln\left|\frac{y-(1+\sqrt{2})}{y-(1-\sqrt{2})}\right|$$

$$= \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-(1+\sqrt{2})}{x^2-(1-\sqrt{2})}\right| \; ; \quad \text{m.x.}$$

$y \in (-1, 1-\sqrt{2})$
$(1-\sqrt{2}, 1+\sqrt{2})$
$(1+\sqrt{2}, +\infty)$

$x \in (-\infty, -\sqrt{1+\sqrt{2}})$
$(-\sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{2}})$
$(\sqrt{1+\sqrt{2}}, +\infty)$.

(3) $\frac{x+1}{x^2+x+1} = \frac{1}{2}\frac{2x+1}{x^2+x+1} + \frac{1}{2}\cdot\frac{1}{x^2+x+1}$;

$$\int \frac{2x+1}{x^2+x+1}dx \left| \begin{array}{l} y = x^2+x+1 \\ dy = (2x+1)dx \end{array} \right\} = \int \frac{dy}{y} = \ln|y|$$

$$= \ln(x^2+x+1) ; \quad x \in \mathbb{R}.$$

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right] ;$$

$$\int \frac{dx}{x^2+x+1} = \frac{4}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} \left| \begin{array}{l} \frac{2x+1}{\sqrt{3}} = y \\ \frac{2}{\sqrt{3}}dx = dy \end{array} \right\} = \frac{2}{\sqrt{3}}\int \frac{dy}{y^2+1}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} y = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) ; \quad x \in \mathbb{R}.$$

m.x.

celkem: $I = \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) ; \quad x \in \mathbb{R}$

(4) $\frac{x^5}{x^2+x-2} = x^3 - x^2 + 3x - 5 + \frac{11x-10}{(x-1)(x+2)}$;

$$\frac{11x-10}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} ; \quad A = \frac{1}{3} ; \quad B = \frac{32}{3}$$

$$I = \int x^3 - x^2 + 3x - 5 + \frac{1}{3}\cdot\frac{1}{x-1} + \frac{32}{3}\cdot\frac{1}{x+2}\,dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{3}{2}x^2 - 5x + \frac{1}{3}\ln|x-1| + \frac{32}{3}\ln|x+2|$$

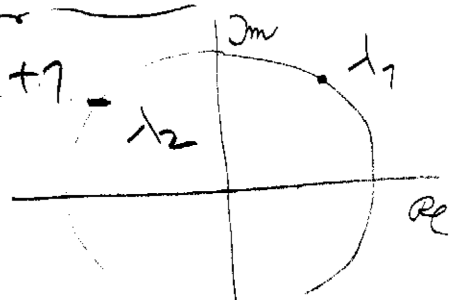
m.x.

$x \in (-\infty, -2)$
$(-2, 1)$
$(1, +\infty)$.

⑤ $x^4+1 = \underbrace{(x-\lambda_1)(x-\bar{\lambda}_1)}_{x^2-\sqrt{2}x+1}\cdot\underbrace{(x-\lambda_2)(x-\bar{\lambda}_2)}_{x^2+\sqrt{2}x+1}$;

$\lambda_1 = \frac{1}{\sqrt{2}}(1+i)$

$\lambda_2 = \frac{1}{\sqrt{2}}(-1+i)$



jiný způsob: $x^4+1 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}x)^2$

$[a^2-b^2=(a+b)(a-b)]$ $\quad = (x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)$

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} ;$$

$A = -C = \frac{1}{2\sqrt{2}}$; $B = D = \frac{1}{2}$ m.x.

$$\frac{\frac{1}{2\sqrt{2}}x \pm \frac{1}{2}}{x^2 \pm \sqrt{2}x+1} = \frac{1}{4\sqrt{2}}\cdot\frac{2x\pm\sqrt{2}}{x^2\pm\sqrt{2}x+1} + \frac{1}{4}\cdot\frac{1}{x^2\pm\sqrt{2}x+1} ;$$

$$\int \frac{2x\pm\sqrt{2}}{x^2\pm\sqrt{2}x+1} = \ln(x^2\pm\sqrt{2}x+1) ; \quad x\in\mathbb{R}$$

$$\int \frac{1}{x^2\pm\sqrt{2}x+1} = \int \frac{2}{(\sqrt{2}x\pm1)^2+1} = \sqrt{2}\cdot \operatorname{arctg}(\sqrt{2}x\pm1) ; \quad x\in\mathbb{R}$$

celkem: $I = \int \frac{dx}{x^4+1} = \int \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} - \frac{\frac{1}{2\sqrt{2}}x-\frac{1}{2}}{x^2-\sqrt{2}x+1}\,dx$

$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x+1)$$

$$+ \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1) ; \quad x\in\mathbb{R}.$$

⑥ $\frac{1}{2}\int \frac{2x}{x^8-1}\,dx \ \left| \begin{array}{l} x^2 = y \\ 2x\,dx = dy \end{array} \right\} = \frac{1}{2}\int \frac{dy}{y^4-1}$

$y^4-1 = (y^2+1)(y^2-1) = (y^2+1)(y+1)(y-1)$

$\frac{1}{y^4-1} = \frac{Ay+B}{y^2+1} + \frac{C}{y+1} + \frac{D}{y-1}$; $A=0$; $B=-\frac{1}{2}$

$C=-\frac{1}{4}$, $D=\frac{1}{4}$

m.x.

$= \frac{1}{2}\int -\frac{1}{2}\cdot\frac{1}{1+y^2} - \frac{1}{4}\cdot\frac{1}{y+1} + \frac{1}{4}\cdot\frac{1}{y-1}\,dy$

$= -\frac{1}{4}\operatorname{arctg} y + \frac{1}{8}\ln\left|\frac{y-1}{y+1}\right|$; $y \in (-\infty,-1)$, $(1,+\infty)$, $(-1,1)$

$= -\frac{1}{4}\operatorname{arctg} x^2 + \frac{1}{8}\ln\left|\frac{x^2-1}{x^2+1}\right|$; $x \in (-\infty,-1)$, $(-1,1)$, $(1,+\infty)$.

m.x.

⑦ $\frac{1}{4}\int \frac{4x^3}{x^8+3}\,dx \ \left| \begin{array}{l} x^4 = y \\ 4x^3 = dy \end{array} \right\} = \frac{1}{4}\int \frac{dy}{y^2+3}$

$= \frac{1}{4\cdot 3}\int \frac{dy}{\left(\frac{y}{\sqrt{3}}\right)^2+1} \ \left| \begin{array}{l} y = \sqrt{3}\,t \\ dy = \sqrt{3}\cdot dt \end{array} \right\} = \frac{1}{4\sqrt{3}}\int \frac{dt}{t^2+1}$

$= \frac{1}{4\sqrt{3}}\operatorname{arctg} t = \frac{1}{4\sqrt{3}}\operatorname{arctg}\left(\frac{x^4}{\sqrt{3}}\right)$; $x \in \mathbb{R}$

m.x.

8. $$\int \frac{x^3}{(x-1)^{100}}\,dx \;\left|\; \begin{array}{l} x-1=y \\ dx=dy \end{array} \right\} = \int \frac{(y+1)^3}{y^{100}}\,dy$$

$$= \int \frac{y^3+3y^2+3y+1}{y^{100}}\,dy = \int y^{-100}+3y^{-99}+3y^{-98}+y^{-97}\,dy$$

$$= -\frac{1}{99y^{99}} - \frac{3}{98y^{98}} - \frac{3}{97y^{97}} - \frac{1}{96y^{96}}, \quad y \in (-\infty,0),\ (0,+\infty)$$

$$= -\frac{1}{99(x-1)^{99}} - \frac{3}{98(x-1)^{98}} - \frac{3}{97(x-1)^{97}} - \frac{1}{96(x-1)^{96}}, \quad x \in (-\infty,1),\ (1,+\infty).$$

9. $$x^5+x^4-2x^3-2x^2+x+1 = (x-1)^2(x+1)^3$$

$$\frac{1}{(x-1)^2(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

$$A=\frac{3}{16},\ B=\frac{1}{4},\ C=\frac{1}{4},\ D=-\frac{3}{16},\ E=\frac{1}{8} \quad \text{m.x.}$$

$$\int \frac{dx}{(x-1)^2(x+1)^3} = \frac{3}{16}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4(x+1)} - \frac{1}{8(x+1)^2} - \frac{1}{8(x-1)}$$

$$x \in (-\infty,-1),\ (-1,+1),\ (1,+\infty)$$

10. $\int \frac{1}{\sqrt{(x-a)(b-x)}}$; notăm $a<b \Rightarrow x \in (a,b)$

$$\frac{1}{\sqrt{(x-a)(b-x)}} = \frac{1}{(x-a)}\sqrt{\frac{x-a}{b-x}} \; ; \quad t = \sqrt{\frac{x-a}{b-x}} \qquad t^2(b-x) = x-a$$

$$t^2 = \frac{x-a}{b-x} \qquad t^2 b - t^2 x = x - a$$

$$x\left(1+t^2\right) = t^2 b + a$$

$$x-a = \frac{t^2 b + \cancel{a} - \cancel{a} - a t^2}{1+t^2} = \frac{t^2(b-a)}{1+t^2} \qquad dx = \frac{-2t(a-b)}{(t^2+1)^2} \qquad x = \frac{t^2 b + a}{1+t^2} = b + \frac{a-b}{t^2+1}$$

$$\to \frac{\cancel{1+t^2}}{\cancel{t^2(b-a)}} \cdot t \cdot \frac{+2\cancel{t}\,\cancel{(a-b)}}{(t^2+1)^{\cancel{2}}}\, dt = \frac{2}{t^2+1} = (2 \operatorname{arctg} t)' \; ;$$

$$\boxed{2 \operatorname{arctg} \sqrt{\frac{x-a}{b-x}}} \checkmark_{m.x.}$$

11. $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\, dx$;

$$\frac{1-\sqrt{\frac{x-1}{x+1}}}{1+\sqrt{\frac{x-1}{x+1}}} \; ; \quad t = \sqrt{\frac{x-1}{x+1}} \; ; \quad t^2 = \frac{x-1}{x+1} \qquad t^2(x+1) = x-1$$

$$t^2 x + t^2 = x - 1$$

$$x = \frac{1+t^2}{1-t^2}$$

$$dx = \frac{2t(1-t^2)+2t(1+t^2)}{(1-t^2)^2} = \frac{4t}{(1-t^2)^2} \checkmark_{m.x.}$$

$$\int \frac{1-t}{1+t} \cdot \frac{4t}{(1-t^2)^2}\, dt = \int \frac{4t}{(1+t)^3(1-t)}\, dt$$

$$\frac{4t}{(1+t)^3(1-t)} = \frac{A}{1-t} + \frac{B}{(1+t)^3} + \frac{C}{(1+t)^2} + \frac{D}{1+t}$$

$$4t = A(1+t)^3 + B(1-t) + C(1-t^2) + D(1+t)^2(1-t^2)$$

$t=1$: $4 = A \cdot 8$; $A = \frac{1}{2}$

$t=-1$: $-4 = 2B$; $B = -2$

$$4 = 3A(1+t)^2 - B - 2Ct + D\left(1-t^2-2t(1+t)\right)$$

$$+D\left(1-2t-3t^2\right)$$

$t=-1$ $\quad 4 = 2 + 2C$; $\quad C = 1$

$2C = 2$

$$0 = 6A(1+t) - 2C + D(-2-6t)$$

$x \in (1, +\infty)$.

$t=-1$: $2C = D(4)$; $D = \frac{1}{2}$ =

$$\frac{4t}{(1+t)^3(1-t)} = \frac{\frac{1}{2}}{1-t} - \frac{2}{(1+t)^3} + \frac{1}{(1+t)^2} + \frac{\frac{1}{2}}{1+t} \quad \text{m.x.}$$

$$= \left(\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + \frac{1}{(1+t)^2} - \frac{1}{1+t}\right) \quad \text{m.x.}$$

$t \in (-\infty, -1)$
$(-1, 1)$
$(1, +\infty)$

12. $\int \sqrt[4]{\frac{x-4}{x+4}}\,dx$; $t = \sqrt[4]{\frac{x-4}{x+4}}$, $t^4 = \frac{x-4}{x+4}$

$t^4x + 4t^4 = x - 4$

$x = \frac{4(1+t^4)}{1-t^4}$

$dx = \frac{32t^3}{(1-t^4)^2}$ m.x.

$$\int \frac{32t^4}{(1-t^4)^2} = I_t^{(1)}$$

$$\frac{32t^4 \pm 32}{(1-t^4)^2} = \frac{32}{t^4-1} + \frac{32}{(1-t^4)^2}$$

I_t

per-partes:

$$\int \frac{dt}{1-t^4} = \frac{t}{1-t^4} - \int \frac{4t^4}{(1-t^4)^2} \quad ; \quad \boxed{\frac{8t}{1-t^4} - 8\int \frac{dt}{1-t^4}}$$

viz č. 6

$u' = 1 \quad u = t$

$v = \frac{1}{1-t^4} \quad v' = \frac{4t^3}{(1-t^4)^2}$

$$\boxed{I_t = \frac{8t}{1-t^4} - 4\,\mathrm{arctg}\, t + 2\ln\left|\frac{t-1}{t+1}\right|}$$

m.x

13. $\int \frac{1-\sqrt{x+1}}{1-\sqrt[3]{x+1}}$; $t = \sqrt[6]{x+1}$, $t^6 = x+1$; $x = t^6 - 1$, $dx = 6t^5\,dt$

$$\int \frac{1-t^3}{1-t^2} \cdot 6t^5\,dt = \int \frac{6t^5(t^2+t+1)}{t+1}\,dt = -6\ln(t+1) + \frac{6}{7}t^7 + \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 - 3t^2 + 6t$$

m.x.

$$\frac{t^5(t^2+t+1)}{t+1} = t^6 + t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}$$

(14) $$\int \frac{x}{\sqrt[4]{x^3(1-x)}}\,dx = \int \sqrt[4]{\frac{x}{1-x}}\,dx \qquad \left|\; t = \sqrt[4]{\frac{x}{1-x}},\quad x = \frac{t^4}{1+t^4}\;;\; dx = \frac{4t^3}{(1+t^4)^2}\right.$$

máme $x \in (0,1)$

m.x.

$$= \int \frac{4t^4}{(t^4+1)^2}\,dt\;;$$

viz: $$\int \frac{dt}{t^4+1} = \frac{t}{t^4+1} + \int \frac{4t^4}{t^4+1}\,dt$$

m.x.

viz č. (5)

(15) $$\int \frac{x\,dx}{(1+x)(1-x^3)\sqrt{\frac{1-x}{1+x}}} \qquad \left|\; t = \sqrt{\frac{1-x}{1+x}}\;;\; x = \frac{1-t^2}{1+t^2}\right.$$

$$dx = \frac{-4t\,dt}{(t^2+1)^2}\;;\quad 1-x^3 = \frac{2t^2(t^4+3)}{(t^2+1)^3}$$

$$1+x = \frac{2}{t^2+1}\,.$$

m.x.

$$= \int \frac{1-t^2}{\cancel{1+t^2}}\cdot\frac{t^2+1}{\cancel{2}}\cdot\frac{\cancel{(t^2+1)^3}}{\cancel{2}t^2(t^4+3)}\cdot\frac{1}{\cancel{t}}\cdot\frac{(-4\cancel{t})}{\cancel{(t^2+1)^2}}\,dt$$

$$= \int \frac{t^4-1}{t^2(t^4+3)}\,dt\;;$$

rozklad: $t^2 = y$: $$\frac{y^2-1}{y(y^2+3)} = -\frac{1}{3y} + \frac{4y}{3(y^2+3)}\;;$$

$$\frac{t^4-1}{t^2(t^4+3)} = \underbrace{-\frac{1}{3t^2}}_{\text{máme}} + \frac{4}{3}\cdot\underbrace{\frac{t^2}{t^4+3}}_{I_1}$$

$$I_1 = \int \frac{t^2}{t^4+3}\,dt = \frac{1}{3}\int \frac{t^2\,dt}{\left(\frac{t}{\sqrt[4]{3}}\right)^4+1} \qquad \left|\; t = x\cdot\sqrt[4]{3},\quad t^2 = x^2\sqrt{3},\quad dt = \sqrt[4]{3}\,dx\right.$$

$$= \frac{1}{3}\cdot\sqrt{3}\cdot\sqrt[4]{3}\int \frac{x^2}{x^4+1}\,dx = \frac{1}{\sqrt[4]{3}}\cdot J\,.$$

pokračování (15): nechť znovu $J = \int \frac{x^2\,dx}{x^4+1}$;

víme (viz (5)): $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

$$\frac{x^2}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \quad m.$$

$$x^2 = (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)$$

$$-A = C = \frac{1}{2\sqrt{2}}\,; \quad B = D = 0. \quad m.x.$$

$$\int \frac{x\,dx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\underbrace{\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx}_{\ln(x^2-\sqrt{2}x+1)} + \frac{1}{\sqrt{2}}\underbrace{\int \frac{dx}{x^2-\sqrt{2}x+1}}_{K}\,;$$

$$K = \int \frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} = \int \frac{2\,dx}{(\sqrt{2}x-1)^2+1} = \frac{2}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1)\,; \quad x\in\mathbb{R}.$$

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tedy: $$\int \frac{x\,dx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\ln(x^2-\sqrt{2}x+1) + \operatorname{arctg}(\sqrt{2}x-1)\,; \quad x\in\mathbb{R}$$

a celkem:

$$J = \frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x+1)$$

$$+ \frac{1}{2\sqrt{2}}\operatorname{arctg}(\sqrt{2}x-1)\,; \quad x\in\mathbb{R}$$

(16) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$

$\sqrt{x^2+x+1} = t - x \qquad x \in \mathbb{R} \Leftrightarrow t \in \left(-\frac{1}{2}, +\infty\right)$

$x^2+x+1 = t^2 - 2tx + x^2 \qquad x=0: \ t=1.$

$x = \dfrac{t^2-1}{2t+1}\ ; \ dx = \dfrac{2(t^2+t+1)}{(2t+1)^2}\,dt$ m.x.

$$\sqrt{x^2+x+1} = t - x = t - \frac{t^2-1}{2t+1} = \frac{t^2+t+1}{2t+1}.$$

$$= \int \frac{2t+1}{t^2-1} \cdot \frac{2t+1}{t^2+t+1} \cdot \frac{2(t^2+t+1)}{(2t+1)^2}\,dt = \int \frac{2\,dt}{t^2-1} = \ln\left|\frac{t-1}{t+1}\right|$$

$t \in (-\infty, -1)$, $(-1, +1)$, $(1, +\infty)$.

celkem: $\displaystyle = \ln\left|\frac{\sqrt{x^2+x+1}+x-1}{\sqrt{x^2+x+1}+x+1}\right|\ ; \quad x \in (-\infty, 0),\ (0, +\infty).$

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(17) $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

$\sqrt{x^2+x+1} = t - x\ ;$

viz již (16) výše:

$$= \int \frac{t^2-1}{2t+1} \cdot \frac{2t+1}{t^2+t+1} \cdot \frac{2(t^2+t+1)}{(2t+1)^2}\,dt = \int \frac{2(t^2-1)}{(2t+1)^2}\,dt$$

$$\frac{2(t^2-1)}{(2t+1)^2} = \frac{1}{2} - \frac{1}{2t+1} - \frac{3}{2(2t+1)^2}\ ;$$ m.x.

$$= \frac{t}{2} - \frac{1}{2}\ln|2t+1| + \frac{3}{4(2t+1)}\ ; \quad t \in \left(-\infty, -\frac{1}{2}\right),\ \left(\frac{1}{2}, +\infty\right).$$

$$= \frac{1}{2}\left(\sqrt{x^2+x+1}+x\right) - \frac{1}{2}\ln\left(2\left(\sqrt{x^2+x+1}+x\right)+1\right)$$

$$- \frac{3}{2} \cdot \frac{1}{\left(2\left(\sqrt{x^2+x+1}+x\right)+1\right)^2}\ ; \quad x \in \mathbb{R}$$

(18) $$\int \frac{x^2+1}{x\sqrt{x^4+1}}\,dx = \frac{1}{2}\int \frac{x^2+1}{x^2\sqrt{(x^2)^2+1}}\cdot 2x\,dx \qquad \Big|\; \begin{array}{l} x^2 = y \\ 2x\,dx = dy \end{array}$$

$$= \frac{1}{2}\int \frac{y+1}{y\sqrt{y^2+1}}\,dy \qquad \Bigg|\; \begin{array}{l} \sqrt{y^2+1} = t - y\;;\; y\in\mathbb{R} \Leftrightarrow t \in (0,+\infty) \\ y^2+1 = t^2 - 2ty + y^2 \qquad \boxed{y=0:\ t=1} \\ y = \dfrac{t^2-1}{2t}\;;\; dy = \dfrac{t^2+1}{2t^2}\,dt \\ y+1 = \dfrac{t^2+2t-1}{2t}\;;\; \sqrt{y^2+1} = \dfrac{t^2+1}{2t} \end{array}$$

$$= \frac{1}{\cancel{2}}\int \frac{t^2+2t-1}{2t}\cdot\frac{\cancel{2t}}{t^2-1}\cdot\frac{\cancel{2t}}{\cancel{t^2+1}}\cdot\frac{\cancel{t^2+1}}{\cancel{2t^2}}\,dt = \int \frac{t^2+2t-1}{2t(t^2-1)}\,dt$$

$$\frac{t^2+2t-1}{2t(t^2-1)} = \frac{1}{2t} + \frac{1}{2(t-1)} - \frac{1}{2(t+1)}\;;\qquad = \frac{1}{2}\ln\left|\frac{t(t-1)}{t+1}\right|$$

$t \in (-\infty,-1)$, $(-1,0)$, $(0,1)$, $(1,+\infty)$.

⇝ výsledek do ~~ryz~~ své neznámé: $x \in (-\infty,0)$, $(0,+\infty)$.

18 - jinak $$\frac{1}{2}\int\frac{y+1}{y\sqrt{y^2+1}} \qquad \Big|\; \begin{array}{l} y = \sinh t \\ dy = \cosh t\,dt \end{array} \quad \sqrt{y^2+1} = \sqrt{\sinh^2 t+1} = \cosh t$$

$$= \frac{1}{2}\int \frac{\sinh t+1}{\sinh t\cdot\cancel{\cosh t}}\cdot\cancel{\cosh t}\,dt = \frac{1}{2}\int\left(1+\frac{1}{\sinh t}\right)dt$$

$$= \int \frac{1}{2}\left(1+\frac{2}{e^t-e^{-t}}\right)dt \qquad \Big|\; \begin{array}{l} e^t = u \qquad dt = \frac{1}{u}\,du \\ t = \ln u \end{array}$$

$$= \int\frac{1}{2}\left(1+\frac{2}{u-\frac{1}{u}}\right)\cdot\frac{du}{u} = \int\frac{u^2+2u-1}{2u(u^2-1)}\,du = \cdots$$ stejný integrál jako výše...

což není metoda, neboť $\sinh^{-1}(R) = \ln\left(\sqrt{R^2+1}+R\right)$.

(21) $\displaystyle\int \frac{dx}{(x^2+1)\sqrt{x^2-1}}$ $\Bigg|$ $\sqrt{x^2-1} = t - x$ $\qquad dx = \frac{t^2-1}{2t^2}\,dt$

$$x^2-1 = t^2 - 2tx + x^2$$

$$x = \frac{t^2+1}{2t} \qquad \sqrt{x^2-1} = \frac{t^2-1}{2t}$$

$$x^2+1 = \frac{t^4+6t^2+1}{4t^2}.$$

$$= \int \frac{4t^2}{t^4+6t^2+1} \cdot \frac{2\cancel{t}}{\cancel{t^2-1}} \cdot \frac{\cancel{t^2-1}}{2\cancel{t^2}}\,dt$$

$$= \int \frac{4t\,dt}{t^4+6t^2+1} \;\left|\; \begin{array}{l} t^2 = u \\ 2t\,dt = du \end{array} \right\} = \int \frac{2\,du}{u^2+6u+1} = \frac{1}{2\sqrt{2}} \ln \left| \frac{u+3-2\sqrt{2}}{u+3+2\sqrt{2}} \right|$$

$$\frac{2}{u^2+6u+1} = \frac{-\frac{1}{2\sqrt{2}}}{u+3+2\sqrt{2}} + \frac{\frac{1}{2\sqrt{2}}}{u+3-2\sqrt{2}} \qquad \text{m-x.}$$

21 - jiný postup: $\quad x = \cosh t\,; \quad t \in (0,+\infty) \Longleftrightarrow x \in (1,+\infty)$

$$dx = \sinh t\,dt\,;$$

$$x^2-1 = \cosh^2 t - 1 = \sinh^2 t\,;$$

$$\text{tj. } \sqrt{x^2-1} = \sinh t\,; \quad t > 0.$$

$$= \int \frac{\sinh t\,dt}{(\cosh^2 t+1\cdot)\sinh t} = \int \frac{dt}{\frac{1}{4}(e^t+e^{-t})^2+1} \;\left|\; \begin{array}{l} e^t = u \\ t = \ln u \\ dt = \frac{du}{u} \end{array} \right.$$

$$= \int \frac{4u}{u^4+6u^2+1} = \dots$$ totéž jako výše;

což se dá čekat, neboť

$$(\sinh)_{-1}(r) = \ln\left(r+\sqrt{r^2-1}\right)$$

$$r \geq 1.$$

(22) $\displaystyle\int \frac{\sqrt{x^2+x+1}}{x^2+2x+1}\,dx$

$\sqrt{x^2+x+1} = t - x$

$x = \dfrac{t^2-1}{2t+1}$; $dx = \dfrac{2(t^2+t+1)}{(2t+1)^2}\,dt$;

$\sqrt{x^2+x+1} = \dfrac{t^2+t+1}{2t+1}$

$x^2+2x+1 = \left(\sqrt{x^2+x+1}\right)^2 + x = (t-x)^2 + x$

$= \dfrac{t^2(t+2)^2}{(2t+1)^2}$; m.x.

$$= \int \frac{t^2+t+1}{\cancel{2t+1}} \cdot \frac{\cancel{(2t+1)^2}}{t^2(t+2)^2} \cdot \frac{2(t^2+t+1)}{(2t+1)^{\cancel{2}}}\,dt = \int \frac{2(t^2+t+1)^2}{t^2(t+2)^2(2t+1)}\,dt$$

$$= \int \frac{2}{2t+1} + \frac{1}{2(t+2)} - \frac{3}{2(t+2)^2} - \frac{1}{2t} + \frac{1}{2t^2}\,dt$$

$$= \ln|2t+1| - \frac{1}{2}\ln|t+2| + \frac{3}{2}\cdot\frac{1}{t+2} - \frac{1}{2}\ln|t| - \frac{1}{2t};$$

-- po resubstituci: $t = x + \sqrt{x^2+x+1}$ platí na $x \in (-\infty, -1)$

$(-1, +\infty)$.

(23) $\displaystyle\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$

$\sqrt{x^2+2x} = t - x$

$\cancel{x^2}+2x = t^2 - 2tx + \cancel{x^2}$ $\qquad dx = \dfrac{t(t+2)}{2(t+1)^2}$

$x = \dfrac{t^2}{2(t+1)}$; $\quad x+1 = \dfrac{t^2+2t+2}{2(t+1)}$; $\sqrt{x^2+2x} = \dfrac{t(t+2)}{2(t+1)}$

$$= \int \frac{32(t+1)^{\cancel{5}\,4}}{(t^2+2t+2)^5} \cdot \frac{\cancel{2(t+1)}}{\cancel{t(t+2)}} \cdot \frac{\cancel{t(t+2)}}{\cancel{2(t+1)^2}}\,dt = \int \frac{32(t+1)^4}{((t+1)^2+1)^5} \quad \left|\begin{array}{l} t+1 = u \\ dt = du \end{array}\right.$$

$$= \int \frac{32u^4}{(u^2+1)^5} = 32\int \frac{1}{(u^2+1)^3} - \frac{2}{(u^2+1)^4} + \frac{1}{(u^2+1)^5}\,du;$$

... při rozkladu značme $u^2 = y$

... k integraci použijeme rekurentní

vzorec pro $I_m = \displaystyle\int \frac{du}{(u^2+1)^m}$; viz př. (48)

123 - jinak: $\sqrt{x(x+2)} = \sqrt{\frac{x+2}{x} x^2} = |x| \sqrt{\frac{x+2}{x}} = \pm x \sqrt{\frac{x+2}{x}}$

$x \in (0, +\infty)$

nebo $(-\infty, -2)$.

$\sqrt{\frac{x+2}{x}} = t \qquad x = \frac{2}{t^2-1} \ \ldots$

$\frac{x+2}{x} = t^2 \qquad dx = \frac{-4t}{(t^2-1)^2} dt \qquad x+1 = \frac{t^2+1}{t^2-1}$

$$\int \frac{dx}{(x+1)^5 \sqrt{x(x+2)}} = \int \frac{dx}{(x+1)^5 x \sqrt{\frac{x+2}{x}}} = \int \frac{(t^2-1)^{\cancel{5}\,4}}{(t^2+1)^5} \cdot \frac{\cancel{t^2-1}}{2} \cdot \frac{(-4t)}{\cancel{(t^2-1)^2}} dt$$

$$= \int \frac{-2t(t^2-1)^4}{(t^2+1)^5} dt \quad \left| \begin{array}{l} t^2 = u \\ 2t\,dt = du \end{array} \right\} = \int \frac{-(u-1)^4}{(u+1)^5} du$$

$$= \int \frac{-1}{u+1} + \frac{8}{(u+1)^2} - \frac{24}{(u+1)^3} + \frac{32}{(u+1)^4} - \frac{16}{(u+1)^5} du$$

$$= -\ln|u+1| - \frac{8}{u+1} + \frac{12}{(u+1)^2} - \frac{32}{3(u+1)^3} + \frac{4}{(u+1)^4} \ \ldots$$