(1) $a_n := \frac{\ln n}{n} x^n$ $\quad$ (*) $\frac{|a_{n+1}|}{|a_n|} = \frac{\ln(n+1)}{\ln(n)} \frac{n+1}{n} |x|$

Protože $\frac{n+1}{n} \to 1$ a $\frac{\ln(n+1)}{\ln n} = \frac{\ln(n) + \ln(1+\frac{1}{n})}{\ln(n)} \to 1$

je (*) $< 1 \iff |x| < 1$. Tudíž $R = 1$.

Řada diverguje pro $|x| > 1$.

Podle růstové škály platí že $\frac{\ln(n)}{n} \to 0$.

Dokažme (skoro) monotonnie:

$$\left(\frac{\ln x}{x}\right)' = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} < 0 \text{ jakmile}$$

$x > e$, tudíž $\frac{\ln(n)}{n} \searrow 0$ pro $n \geq 3$.

$x^n = e^{i\theta n}$ pro $|x| = 1$. $e^{i\theta n}$ má omezené částečné součty pro $\theta \in (0, 2\pi)$ t.j $x \neq 1$.

Dle Dirichleta $\sum a_n$ konverguje jakmile

$|x| = 1$, $x \neq 1$.

Návody na A

(2) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ – standardní výpočet na podílové kritérium

$R=4.$

$$4^n \frac{(n!)^2}{(2n)!} = \frac{2^n\, n!}{1 \cdots (2n-3)(2n-1)} = \frac{1}{(1-\frac{1}{2n})(1-\frac{1}{2(n-1)}) \cdots (\frac{1}{2})}$$

$$= e^{-\sum_{k=1}^{n} \ln(1-\frac{1}{2k})} \approx e^{C+\sum_{k=k_0}^{n} \frac{1}{2k}} = \infty.$$

Není splněná nutná podmínka na kruhu.

(3) Standardní $R=5$ a Dirichlet na $|x-3|=5$ kromě $x=8$

(4) Odmocninové kritérium

$$|a^n x| < 0 \Leftrightarrow a \in (0,1)$$

žádná konvergence na kruhu – není splněná nutná podmínka

(5) BÚNO $|a| \geq |b|$ pak $R = \frac{1}{|a|}$ Pokud $|a| > |b|$ Na kruhu dle Dirichleta a abs kon členu s b^n.

⑤ podm.

pokud $a=b$ Dirichlet.

pokud $a=-b$ také Dirichlet na kruhu.

⑥ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{(n^2)} x^n$ odmocninové krit $\Rightarrow R=\frac{1}{e}$

$$\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n = e^{n\left(\ln\left(1+\frac{1}{n}\right)^n - 1\right)}$$

$$= e^{n^2\left(\ln\left(1+\frac{1}{n}\right)-1\right)}$$

$$\ln(1+t) = 1 + t - \frac{t^2}{2} + o(t^2)$$

$$\ln\left(1+\frac{1}{n}\right) - 1 = 1 - 1 + \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$$

$$n^2\left(\ln\left(1+\frac{1}{n}\right)-1\right) = n + o(n) \xrightarrow{n\to\infty} \infty$$

Není splněná nutná podmínka!

⑦ Podobně jak ②.

B

① Najděte a_n: $\sum_{n=0}^{\infty} a_n x^n = \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$ tudíž $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \cdot 4^n$$

$$\sin^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n} 4^n}{2 \cdot (2n)!} \quad R = \infty.$$

② $\left(\sqrt{1+t}\right)''' = \left(\frac{1}{2}(1+t)^{-\frac{1}{2}}\right)'' = \left(-\frac{1}{4}(1+t)^{-\frac{3}{2}}\right)' = \frac{3}{8}(1+t)^{-\frac{5}{2}}$

$\left((1+t)^{\frac{1}{2}}\right)^{(n)} = (-1)^{n+1} \frac{(2n-3)!!}{2^n}$ kde $(2n-3)!! = (2n-3)(2n-5)(2n-7)\ldots 3 \cdot 1$

$$\sqrt{1+t} = 1 + \frac{1}{2}t + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{(2n-3)!!\, t^n}{2^n n!}$$

a

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n+1} \frac{(2n-3)!!}{2^n n!} x^{2n}$$

$R = 1.$

(3)

$$\text{arctg}\, t = \sum a_n t$$

$$\text{arctg}\, t = \int \frac{1}{1+t^2}\, dt \qquad \frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty} (-t)^n$$

$$\rightarrow \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad \text{na } (-1,1)$$

$$\Rightarrow \int \frac{1}{1+t^2}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{2n+1} = \text{arctg}\, t \quad \text{na } (-1,1)$$

$$\int \frac{\text{arctg}\, t}{t}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)^2} + 1. \quad \text{na } (-1,1).$$

C 1 $\displaystyle\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\sum n x^n\right)'$

$R=1$

$$= \left(x \sum n x^{n-1}\right)'$$

$$= \left(x \left(\sum x^n\right)'\right)'$$

$$= \left(x \left(\frac{1}{1-x}\right)'\right)'$$

$$= \left(\frac{x}{(1-x)^2}\right)' = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

D 1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,2^n} = \left(\sum_{n=1}^{\infty} \frac{1}{n} x^n\right)_{x=\frac{1}{2}}$

$$\left(\sum_{n=1}^{\infty} \frac{1}{n} x^n\right)' = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n} = \int \frac{1}{1-x}\,dx + C = -\ln(1-x)$$

$$x=\frac{1}{2} \quad \sum_{n=1}^{\infty} \frac{1}{n2^n} = -\ln\frac{1}{2} = \ln 2 .$$

2) $$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \left(\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}\right)_{x=1}$$

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} = x \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{n!} = x\left(\sum_{n=1}^{\infty} \frac{n x^n}{n!}\right)'$$

$$= x\left(x\left(\sum_{n=1}^{\infty} \frac{x^n}{n!}\right)'\right)' = x\left(x e^x\right)' = x e^x + x^2 e^x$$

$x = 1$ $$\sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e.$$

Též: $$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(n+1)}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n (n+1)}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\right)' = \left(x e^x\right)'$$

$$= x e^x + e^x$$ pokud $x = 1$

Výsledek $\sum \frac{n^2}{n!} = 2e.$

(3) Stačí rozvinout ovyty x v řadě a položit $x=1$.

(4) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = \left(\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)\cdot n} \right)_{x=1}$$

$$\int \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \circledast = \iint \sum_{n=1}^{\infty} (-1)^n x^{n-1} = \iint \sum_{n=0}^{\infty} (-1)^{n+1} x^n$$

$$= -\iint \frac{1}{1+x} = -\int \ln(1+x) + C$$

$C=0$ $\circledast$ has 0 term $=0$.

$$\int \ln(1+x) = x \ln(1+x) - \int \frac{x}{1+x}$$

$$= x \ln(1+x) - \int 1 - \frac{1}{1+x}$$

$$= x \ln(1+x) - x + \ln(1+x) \underset{\text{na } C}{\overset{\text{až}}{=}} (1+x)\ln(1+x) - x$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)\cdot n} = (1+x)\ln(1+x) - x$$ protože LHS $=0$ v $x=0$.

polož $x=1$ $\Rightarrow$ $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = 2\ln 2 - 1.$$