Challenges and Opportunities in Mixed Precision Numerical Linear Algebra

Erin Carson
Charles University

ICL Lunch Talk
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Floating Point Formats

\[ (-1)^{\text{sign}} \times 2^{(\text{exponent} - \text{offset})} \times 1.\text{fraction} \]

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Hardware Support for Multiprecision Computation

Use of low precision in machine learning has driven emergence of low-precision capabilities in hardware:

- Half precision (FP16) defined as storage format in 2008 IEEE standard
- ARM NEON: SIMD architecture, instructions for 8x16-bit, 4x32-bit, 2x64-bit
- AMD Radeon Instinct MI25 GPU, 2017:
  - single: 12.3 TFLOPS, half: 24.6 TFLOPS
- NVIDIA Tesla P100, 2016: native ISA support for 16-bit FP arithmetic
- NVIDIA Tesla V100, 2017: tensor cores for half precision;
  - 4x4 matrix multiply in one clock cycle
  - double: 7 TFLOPS, half+tensor: 112 TFLOPS (16x!)
- Google's Tensor processing unit (TPU)
- NVIDIA A100, 2020: tensor cores with multiple supported precisions: FP16, FP64, Binary, INT4, INT8, bfloat16
- NVIDIA H100, 2022: now with quarter-precision (FP8) tensor cores
- Future exascale supercomputers: (~2021) Expected extensive support for reduced-precision arithmetic (32/16/8-bit)
Mixed precision in NLA

- **BLAS**: cuBLAS, MAGMA, [Agullo et al. 2009], [Abdelfattah et al., 2019], [Haidar et al., 2018]
- **Iterative refinement**:
  - Long history: [Wilkinson, 1963], [Moler, 1967], [Stewart, 1973], ...
  - More recently: [Langou et al., 2006], [C., Higham, 2017], [C., Higham, 2018], [C., Higham, Pranesh, 2020], [Amestoy et al., 2021]
- **Matrix factorizations**: [Haidar et al., 2017], [Haidar et al., 2018], [Haidar et al., 2020], [Abdelfattah et al., 2020]
- **Eigenvalue problems**: [Dongarra, 1982], [Dongarra, 1983], [Tisseur, 2001], [Davies et al., 2001], [Petschow et al., 2014], [Alvermann et al., 2019]
- **Sparse direct solvers**: [Buttari et al., 2008]
- **Orthogonalization**: [Yamazaki et al., 2015]
- **Multigrid**: [Tamstorf et al., 2020], [Richter et al., 2014], [Sumiyoshi et al., 2014], [Ljungkvist, Kronbichler, 2017, 2019]
- **(Preconditioned) Krylov subspace methods**: [Emans, van der Meer, 2012], [Yamagishi, Matsumura, 2016], [C., Gergelits, Yamazaki, 2021], [Clark, 2019], [Anzt et al., 2019], [Clark et al., 2010], [Gratton et al., 2020], [Arioli, Duff, 2009], [Hogg, Scott, 2010]

For survey and references, see [Abdelfattah et al., IJHPC, 2021]
HPL-AI Benchmark

- Like HPL, solves dense $Ax=b$, results still to double precision accuracy
- Achieves this via mixed-precision iterative refinement
## HPL-AI Benchmark

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  • Error bound with constant $nu$ provides no information if $nu > 1$
  • One solution: probabilistic approach [Higham, Mary, 2019], [Higham, Mary, 2020]
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• Smaller range of representable numbers
  • Limited range of lower precision might cause overflow when rounding
  • Quantities rounded to lower precision may lose important numerical properties (e.g., positive definiteness)
  • One solution: scaling and shifting approach [Higham, Pranesh, 2019]
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• Larger unit roundoff
  • Lose something small when storing: $fl(x) = x(1 + \delta), \ |\delta| \leq u$
  • Lose something small when computing: $fl(x \ op \ y) = (x \ op \ y)(1 + \delta), \ |\delta| \leq u$
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Does it matter?
Inexact computations

- In real computations we have many sources of inexactness
  - Imperfect data, measurement error
  - Modeling error, discretization error
  - Intentional approximation to improve performance
    - Reduced models, Low-rank representations, sparsification, randomization

[Schilders, van der Vorst, Rommes, 2008]
Low-rank (hierarchical) approximation
Sinha, 2018

Sparsification, Randomized algorithms
[Sinha, 2018]
Inexact computations

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• Given that we are already working with so much inexactness, does it matter if we use lower precision?
  • Analysis of accuracy in techniques that use intentional approximation almost always assume that roundoff error is small enough to be ignored
  • Is this true? Is it true even if we use low precision?
Example: Randomized Algorithms

• Given $m \times n$ $A$, want truncated SVD with parameter $k$

$A \approx \hat{U}\hat{\Sigma}\hat{V}^T$
Example: Randomized Algorithms

• Given $m \times n$ $A$, want truncated SVD with parameter $k$

\[
\begin{align*}
A & \approx \hat{U} \hat{\Sigma} \hat{V}^T \\
\end{align*}
\]

• Randomized SVD:

\[
\begin{align*}
A \Omega &= Q R = Q^T A = B = \tilde{U} \tilde{\Sigma} \tilde{V}^T \\
\end{align*}
\]

Assuming exact arithmetic:
If $Q$ satisfies $\|A - QQ^T A\| \leq \varepsilon$, then $\|A - \tilde{U} \tilde{\Sigma} \tilde{V}^T\| \leq \varepsilon$
What happens in finite precision?

Let's try different types of randsvd matrices from the MATLAB gallery:

\[
A = \text{gallery('randsvd',[100,40],1e6,\text{mode}); \ k=15;}
\]

\[
[U,S,V] = \text{svd}(A) : \text{non-randomized SVD, exact arithmetic}
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\[
[\hat{U},\hat{S},\hat{V}] = \text{rsvd}(A) : \text{randomized SVD, exact arithmetic}
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\[
[\hat{U}_d,\hat{S}_d,\hat{V}_d] = \text{rsvd}(A) : \text{randomized SVD, double precision}
\]

\[
[\hat{U}_h,\hat{S}_h,\hat{V}_h] = \text{rsvd}(A) : \text{randomized SVD, half precision}
\]
What happens in finite precision?

Let's try different types of randsvd matrices from the MATLAB gallery:

\[
A = \text{gallery('randsvd',[100,40],1e6,\texttt{mode}); } \text{k=15;}
\]

\[
[U,S,V] = \text{svd}(A) : \text{non-randomized SVD, exact arithmetic}
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[\tilde{U},\tilde{S},\tilde{V}] = \text{rsvd}(A) : \text{randomized SVD, exact arithmetic}
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\]

\[
[\tilde{U}_h,\tilde{S}_h,\tilde{V}_h] = \text{rsvd}(A) : \text{randomized SVD, half precision}
\]

Mode 3: Geometrically distributed singular values

\[
\|A - USV^T\|_2 = 4.92\cdot10^{-3}
\]

\[
\|A - \tilde{U}\tilde{S}\tilde{V}^T\|_2 = 4.92\cdot10^{-3}
\]

\[
\|A - \tilde{U}_d\tilde{S}_d\tilde{V}_d^T\|_2 = 4.92\cdot10^{-3}
\]

\[
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**Mode 3: Geometrically distributed singular values**

\[
\|A - USV^T\|_2 = 4.92e-03
\]

\[
\|A - \tilde{U} \tilde{S} \tilde{V}^T\|_2 = 4.92e-03
\]

\[
\|A - \tilde{U}_d \tilde{S}_d \tilde{V}_d^T\|_2 = 4.92e-03
\]

\[
\|A - \tilde{U}_h \tilde{S}_h \tilde{V}_h^T\|_2 = 4.92e-03
\]

**Mode 1: one large singular value**

\[
\|A - USV^T\|_2 = 1.00e-06
\]

\[
\|A - \tilde{U} \tilde{S} \tilde{V}^T\|_2 = 1.17e-06
\]

\[
\|A - \tilde{U}_d \tilde{S}_d \tilde{V}_d^T\|_2 = 1.17e-06
\]

\[
\|A - \tilde{U}_h \tilde{S}_h \tilde{V}_h^T\|_2 = 1.11e-05
\]
What happens in finite precision?

Let’s try different types of randsvd matrices from the MATLAB gallery:

```matlab
A = gallery('randsvd',[100,40],1e6,mode); k=15;
```

\[
[U,S,V] = \text{svd}(A) : \text{non-randomized SVD, exact arithmetic}
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Use of low precision leads to an order magnitude loss of accuracy! Roundoff error can’t be ignored!
What happens in finite precision?

Let’s try different types of randsvd matrices from the MATLAB gallery:

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\]

\[
[\hat{U}_h,\hat{S}_h,\hat{V}_h] = \text{rsvd}(A): \text{randomized SVD, half precision}
\]

Mode 3: Geometrically distributed singular values

\[
\|A - USV^T\|_2 = 4.92\times 10^{-3}
\]

\[
\|A - \hat{U}\hat{S}\hat{V}^T\|_2 = 4.92\times 10^{-3}
\]

\[
\|A - \hat{U}_d\hat{S}_d\hat{V}_d^T\|_2 = 4.92\times 10^{-3}
\]

\[
\|A - \hat{U}_h\hat{S}_h\hat{V}_h^T\|_2 = 4.92\times 10^{-3}
\]

Mode 1: one large singular value

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\[
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\[
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\]

\[
\|A - \hat{U}_h\hat{S}_h\hat{V}_h^T\|_2 = 1.11\times 10^{-5}
\]

\[
\|A - Q_hQ_h^TA\|_2 = 3.59\times 10^{-6}
\]

Error bound no longer holds!

Use of low precision leads to an order magnitude loss of accuracy! Roundoff error can’t be ignored!
• Block low-rank approximation and hierarchical matrix representations arise in a variety of applications

• Work on mixed and low precision in block low-rank computations

• [Higham, Mary, 2019]: block low-rank LU factorization preconditioner that exploits numerically low-rank structure of the error for LU computed in low precision

• [Higham, Mary, 2019]: Interplay of roundoff error and approximation error in solving block low-rank linear systems using LU

• [Buttari, et al., 2020]: block low-rank single precision coarse grid solves in multigrid

• [Amestoy et al., 2021]: Mixed precision low rank approximation and application to block low-rank LU factorization
Inverse multiquadratic kernel:

\[
A(i,j) = \frac{1}{\sqrt{1 + 0.1\|x - y\|^2}}, \quad x, y \in \mathbb{R}^2
\]

A is SPD. Low-rank approximation of A should also be SPD!
Example: Low-Rank Approximation

Inverse multiquadratic kernel:

\[ A(i, j) = \frac{1}{\sqrt{1 + 0.1\|x - y\|^2}}, \quad x, y \in \mathbb{R}^2 \]

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Exact arithmetic SVD:
Example: Low-Rank Approximation

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Half precision SVD:
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Example: Low-Rank Approximation

Inverse multiquadratic kernel:

\[ A(i, j) = \frac{1}{\sqrt{1 + 0.1 \|x - y\|^2}}, \quad x, y \in \mathbb{R}^2 \]

A is SPD. Low-rank approximation of A should also be SPD!

Exact arithmetic SVD:

Half precision SVD:

Positive definiteness lost!
Example: Iterative Methods

\[ A = \text{diag}(\text{linspace}(0.001,1,100)); \]
\[ [V,\sim] = \text{eig}(A); \]
\[ b = V'*\text{ones}(n,1); \]
Example: Iterative Methods

\[ n = 100, \lambda_1 = 10^{-3}, \lambda_n = 1 \]

\[ \lambda_i = \lambda_1 + \left( \frac{i-1}{n-1} \right) (\lambda_n - \lambda_1) (0.65)^{n-i}, \quad i = 2, ..., n-1 \]

\[ [V, \sim] = \text{eig}(A); \]

\[ b = V' \ast \text{ones}(n,1); \]
• Low precision can have massive performance benefits but must be used with caution!

• Many opportunities for using mixed and low precision computation in scientific applications

• Need to develop a theoretical understanding of how mixed precision algorithms behave; need to revisit analyses of algorithms and techniques that ignore finite precision
Iterative Refinement for $Ax = b$

Iterative refinement: well-established method for improving an approximate solution to $Ax = b$

$A$ is $n \times n$ and nonsingular; $u$ is unit roundoff

Solve $Ax_0 = b$ by LU factorization

for $i = 0$: maxit

$r_i = b - Ax_i$  \hspace{1cm} (in precision $u_f$)

Solve $Ad_i = r_i$  \hspace{1cm} (in precision $u_r$)

$x_{i+1} = x_i + d_i$  \hspace{1cm} (in precision $u_s$)

(in precision $u$)
Iterative Refinement in 3 Precisions

- 3-precision iterative refinement [C. and Higham, 2018]

\[ u_f = \text{factorization precision}, \quad u = \text{working precision}, \quad u_r = \text{residual precision} \]

\[ u_f \geq u \geq u_r \]

\( u_s \) is the effective precision of the solve, with \( u \leq u_s \leq u_f \)

- For triangular solves with LU factors: \( u_s = u_f \)
- For GMRES preconditioned by LU factors, \( u_s = u \) [C. and Higham, 2017]

- New analysis **generalizes** existing types of IR:

<table>
<thead>
<tr>
<th>Traditional</th>
<th>( u_f = u, u_r = u^2 )</th>
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</thead>
<tbody>
<tr>
<td>Fixed precision</td>
<td>( u_f = u = u_r )</td>
</tr>
<tr>
<td>Lower precision factorization</td>
<td>( u_f^2 = u = u_r )</td>
</tr>
</tbody>
</table>

- Enables **new** types of IR: (half, single, double), (half, single, quad), (half, double, quad), etc.
Standard (LU-based) IR in three precisions ($u_s = u_f$)

Half $\approx 10^{-4}$, Single $\approx 10^{-8}$, Double $\approx 10^{-16}$, Quad $\approx 10^{-34}$

<table>
<thead>
<tr>
<th>$u_f$</th>
<th>$u$</th>
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## IR3: Summary

Standard (LU-based) IR in three precisions ($u_s = u_f$)

Half $\approx 10^{-4}$, Single $\approx 10^{-8}$, Double $\approx 10^{-16}$, Quad $\approx 10^{-34}$

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Standard (LU-based) IR in three precisions ($u_s = u_f$)

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**IR3: Summary**

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## IR3: Summary

Standard (LU-based) IR in three precisions ($u_s = u_f$)

Half $\approx 10^{-4}$, Single $\approx 10^{-8}$, Double $\approx 10^{-16}$, Quad $\approx 10^{-34}$

<table>
<thead>
<tr>
<th></th>
<th>$u_f$</th>
<th>$u$</th>
<th>$u_r$</th>
<th>$\max k_\infty(A)$</th>
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⇒ Benefit of IR3 vs. "LP fact.": no $\text{cond}(A, x)$ term in forward error
Standard (LU-based) IR in three precisions ($u_s = u_f$)

Half $\approx 10^{-4}$, Single $\approx 10^{-8}$, Double $\approx 10^{-16}$, Quad $\approx 10^{-34}$

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$\Rightarrow$ Benefit of IR3 vs. traditional IR: As long as $\kappa_\infty(A) \leq 10^4$, can use lower precision factorization w/no loss of accuracy!
### GMRES-IR: Summary

GMRES-IR: Solve for $d_i$ via GMRES on $U^{-1}L^{-1}Ad_i = U^{-1}L^{-1}r_i$

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<tr>
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⇒With GMRES-IR, lower precision factorization will work for higher $\kappa_\infty(A)$
GMRES-IR: Summary

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$\Rightarrow$ With GMRES-IR, lower precision factorization will work for higher $\kappa_{\infty}(A)$
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$\Rightarrow$ As long as $\kappa_\infty(A) \leq 10^{12}$, can use half precision factorization and still obtain double precision accuracy!

$\kappa_\infty(A) \leq u^{-1/2} u_f^{-1}$
**GMRES-IR: Summary**

GMRES-IR: Solve for $d_i$ via GMRES on $U^{-1}L^{-1}Ad_i = U^{-1}L^{-1}r_i$

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Recent work: 5-precision GMRES-IR [Amestoy, et al., 2021]
GMRES-IR: Summary

GMRES-IR: Solve for $d_i$ via GMRES on $U^{-1}L^{-1}Ad_i = U^{-1}L^{-1}r_i$

GMRES-based IR in three precisions ($u_s = u$)

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Recent work: 5-precision GMRES-IR [Amestoy, et al., 2021]
• Want to solve

$$\min_x \|b - Ax\|_2$$

where $A \in \mathbb{R}^{m \times n}$ ($m > n$) has rank $n$

• Commonly solved using QR factorization:

$$A = QR = [Q_1, Q_2] \begin{bmatrix} U \\ 0 \end{bmatrix}$$

where $Q$ is an $m \times m$ orthogonal matrix and $U$ is upper triangular.

$$x = U^{-1}Q_1^Tb, \quad \|b - Ax\|_2 = \|Q_2^Tb\|_2$$
Extension: Least Squares Problems

- Want to solve
  \[
  \min_x \| b - Ax \|_2
  \]
  where \( A \in \mathbb{R}^{m \times n} \) (\( m > n \)) has rank \( n \)

- Commonly solved using QR factorization:
  \[
  A = QR = [Q_1, Q_2] \begin{bmatrix} U \\ 0 \end{bmatrix}
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  where \( Q \) is an \( m \times m \) orthogonal matrix and \( U \) is upper triangular.
  \[
  x = U^{-1} Q_1^T b, \quad \| b - Ax \|_2 = \| Q_2^T b \|_2
  \]

- As in linear system case, for ill-conditioned problems, iterative refinement often needed to improve accuracy and stability
Extension: Least Squares Problems

• For inconsistent systems, must simultaneously refine both solution and residual

• (Björck, 1967): Least squares problem can be written as a linear system with square matrix of size $(m + n)$:

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$
• For inconsistent systems, must simultaneously refine both solution and residual

• (Björck, 1967): Least squares problem can be written as a linear system with square matrix of size \((m + n)\):

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]

• Refinement proceeds as follows:

1. Compute "residuals"

\[
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0
\end{bmatrix}
- 
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r_i \\
x_i
\end{bmatrix}
= 
\begin{bmatrix}
b - r_i - Ax_i \\
-A^T r_i
\end{bmatrix}
\]

2. Solve for corrections

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix}
= 
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix}
\]

3. Update "solution":

\[
\begin{bmatrix}
r_{i+1} \\
x_{i+1}
\end{bmatrix}
= 
\begin{bmatrix}
r_i \\
x_i
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix}
\]
For inconsistent systems, must simultaneously refine both solution and residual

(Björck, 1967): Least squares problem can be written as a linear system with square matrix of size \((m + n)\):

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0
\end{bmatrix}
\quad \tilde{A}\tilde{x} = \tilde{b}
\]

Refinement proceeds as follows:

1. Compute "residuals"
   \[
   \begin{bmatrix}
   f_i \\
g_i
   \end{bmatrix}
   = 
   \begin{bmatrix}
b \\
0
\end{bmatrix} - 
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r_i \\
x_i
\end{bmatrix}
= 
\begin{bmatrix}
b - r_i - Ax_i \\
-A^T r_i
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Extension: Least Squares Problems

- For inconsistent systems, must simultaneously refine both solution and residual.
- (Björck, 1967): Least squares problem can be written as a linear system with square matrix of size \((m + n)\):

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta r \\
\Delta x
\end{bmatrix}
= 
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix}
\]

\(\tilde{A}\tilde{x} = \tilde{b}\)

- Refinement proceeds as follows:

1. Compute "residuals"

\[
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix} = 
\begin{bmatrix}
b \\
0
\end{bmatrix} - 
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix} = 
\begin{bmatrix}
b - r_i - Ax_i \\
-A^T r_i
\end{bmatrix}
\]

\(\tilde{r}_i = \tilde{b} - \tilde{A}\tilde{x}_i\)

2. Solve for corrections

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix}
= 
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix}
\]

\(\tilde{A}d_i = \tilde{r}_i\)

3. Update "solution":

\[
\begin{bmatrix}
\tilde{x}_{i+1} \\
\tilde{x}_i + d_i
\end{bmatrix} = 
\begin{bmatrix}
\tilde{x}_i \\
\Delta x_i
\end{bmatrix}
\]
Extension: Least Squares Problems

• For inconsistent systems, must simultaneously refine both solution and residual
• (Björck,1967): Least squares problem can be written as a linear system with square matrix of size \((m + n)\):

\[
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r \\
x
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix}
\]

\[\tilde{A}\tilde{x} = \tilde{b}\]

• Refinement proceeds as follows:
1. Compute "residuals"
   \[
   \begin{bmatrix}
f_i \\
g_i
\end{bmatrix} =
\begin{bmatrix}
b \\
0
\end{bmatrix} -
\begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
r_i \\
x_i
\end{bmatrix} =
\begin{bmatrix}
b - r_i - Ax_i \\
-\tilde{A}^T r_i
\end{bmatrix}
\]

2. Solve for corrections
   \[
   \begin{bmatrix}
I & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix} =
\begin{bmatrix}
f_i \\
g_i
\end{bmatrix}
\]

3. Update "solution":
   \[
   \begin{bmatrix}
r_{i+1} \\
x_{i+1}
\end{bmatrix} =
\begin{bmatrix}
r_i \\
x_i
\end{bmatrix} +
\begin{bmatrix}
\Delta r_i \\
\Delta x_i
\end{bmatrix}
\]

Results for 3-precision IR for linear systems also applies to least squares problems [C., Higham, Pranesh, 2020]
Extension: Multistage Mixed Precision IR

- Many different variants of mixed precision IR
  - “standard IR” (SIR): LU solves
  - SGMRES-IR: preconditioned GMRES entirely in working precision
  - GMRES-IR: preconditioned GMRES with extra precision
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  • Requires recomputing the expensive LU factorization in higher precision
  • Before resorting to increasing precisions, we can first try using a better inner solver with the existing LU factors!
Many different variants of mixed precision IR

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→ Multistage Iterative Refinement (MSIR) [Oktay, C., NLAA, 2022]
well-conditioned:
same as SIR
Switch to SGMRES-IR for medium-conditioned problems:
ill-conditioned: switch twice to GMRES-IR
Extension: SPAI-GMRES-IR

- Existing analyses of GMRES-IR assume we use full LU factors
- In practice, often want to use sparse preconditioners (ILU, SPAI, etc.)
- [C., Khan, arXiv:2202.10204, 2022]: analysis of GMRES-IR with SPAI preconditioning
  - Convergence to limiting accuracy as long as \( nu_f \text{cond}(A^T) \lesssim n\varepsilon \lesssim u^{-1/2} \)
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Matrix: steam1, $(u_f, u, n_r) = (\text{single, double, quad})$

- $\text{nnz}(L + U) = 21,657$
- $\text{nnz}(M) = 2,248$
The rise of multiprecision hardware

- Future machines will support a range of precisions: quarter, half, single, double, quad
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  - e.g., bfloat16 (truncated 16-bit version of single precision), posits
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• As numerical analysts, we must determine when and where we can exploit lower-precision hardware to improve performance
Thank you!
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