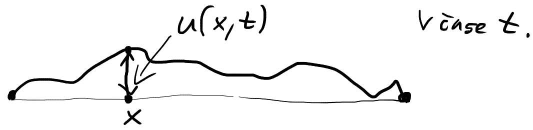
Fourierovy řady

Motivační úloha: rovnice struny (vlnová rovnice v 1D)

mějme strunu upevněnou na koncích 

vychýlíme ji do předepsaného tvaru a v čase t=0 pustíme

jak se bude chovat?

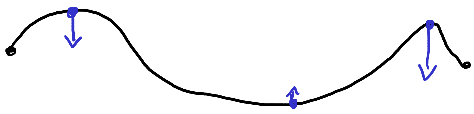


Hledáme $u : \underbrace{[0,L]}_{\text{struna délky } L} \times \underbrace{[0,\infty)}_{\text{čas, od teď do budoucna}} \to \mathbb{R}$,

splňující pro $x \in (0,L)$, $t \in (0,\infty)$:, $a \geq 0$.

(1) $$\underbrace{\frac{\partial^2 u}{\partial t^2}(x,t)}_{\text{zrychlení bodu } x} - a^2 \underbrace{\frac{\partial^2 u}{\partial x^2}(x,t)}_{\text{zahnutí struny v bodě } x} = 0$$

(a čase t)



"zahnutá struna se chce narovnat"

(2) $$\begin{cases} u(0,t) = u(L,t) = 0, & \text{upevněná na koncích} \\ u(x,0) = g(x) & \text{v čase 0 má tvar } g : [0,L] \to \mathbb{R}. \\ \frac{\partial u}{\partial t}(x,0) = 0 & \text{v čase 0 nulová rychlost} \end{cases}$$

Řešení Pokud $g(x) = \sin\left(\frac{\pi}{L} n x\right)$, pak

řešení je $u(x) = \sin\left(\frac{\pi}{L} n x\right) \cos\left(\frac{\pi a}{L} n \cdot t\right)$.

Co pro obecné g? Pokud$^{(?)}$ lze psát $g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n x\right)$, ← Fourierova řada.

můžeme zkusit položit $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n x\right) \cos\left(\frac{\pi a}{L} n t\right)$

Pak, pokud$^{(?)}$ lze prohodit a $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial t^2}$, splňuje u (1)

((2) u také splňuje.)

Konvergence Fourierových řad

Úloha: zadanou funkci $f:(-\pi,\pi)\to\mathbb{R}$ zapsat ve tvaru Fourierovy řady

Komplexní tvar: $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$, $\quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-int}\,dt$, $n\in\mathbb{Z}$

Reálný tvar: $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$,

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$, $\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$, $\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$, $\quad n\in\mathbb{N}$

platí jen za určitých předpokladů (*)

Vztah mezi a_n, b_n, c_n: $\quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$, $\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i}$

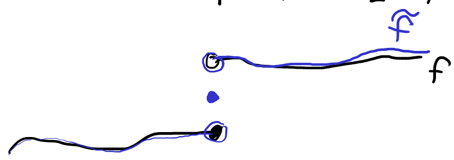
$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\frac{e^{inx}+e^{-inx}}{2}+b_n\frac{e^{inx}-e^{-inx}}{2i}\right)=$$

$$=\underbrace{\frac{1}{2}a_0}_{c_0}+\sum_{n=1}^{\infty}\left(\underbrace{\frac{a_n-ib_n}{2}}_{c_n}e^{inx}+\underbrace{\frac{a_n+ib_n}{2}}_{c_{-n}}e^{-inx}\right),\quad n\in\mathbb{N}$$

(*): **Jordan-Dirichletovo kritérium**

- $f\in BV([-\pi,\pi])\cap \mathcal{C}([-\pi,\pi]) \implies \sum_{n=-\infty}^{\infty} c_n e^{inx} \overset{loc}{\rightrightarrows} f$ na $(-\pi,\pi)$.
- Pouze $f\in BV([-\pi,\pi]) \implies \sum_{n=-\infty}^{\infty} c_n e^{inx} \to \tilde{f}$ na $[-\pi,\pi]$,

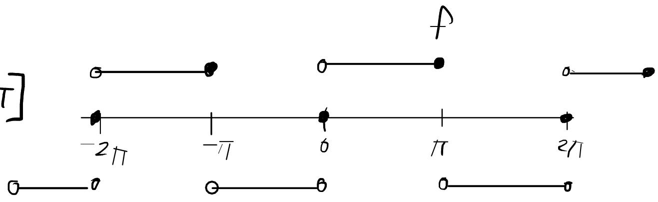
kde $\tilde{f}(x)=\dfrac{\lim_{t\to x^+} f(t)+\lim_{t\to x^-} f(t)}{2}$



(Víme, že $\tilde{f}=f$ na $[-\pi,\pi]\setminus S$ spočetná)

Pozn. kdyby pouze $f\in\mathcal{C}([-\pi,\pi])$, $f\notin BV([-\pi,\pi])$, máme pouze Cesàrovskou konvergenci Fourierovy řady (16.5 Fejér)

$\boxed{1}$ $f(x) = \operatorname{sgn} x, \quad x \in (-\pi, \pi).$ Dodefinujte f 2π-periodicky na $\mathbb{R}$, rozvinte do Fourierovy řady, rozhodněte, zda řada konverguje k f.



Dodefinování na $\mathbb{R}$:

$$f(x) = \operatorname{sgn}(x - 2\ell\pi), \quad x \in \big((2\ell-1)\pi, (2\ell+1)\pi\big]$$

$$f(x) = \begin{cases} 1, & x \in (2\ell\pi, (2\ell+1)\pi] \\ 0, & x = 2\ell\pi \\ -1, & x \in ((2\ell-1)\pi, 2\ell\pi) \end{cases}$$

Spočítáme Fourierovy koeficienty:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} t \, dt \xleftarrow{\text{sgn lichá, } (-\pi,\pi) \text{ symetrický kolem } 0} = 0$$

$$n \geq 1: \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} t \cos nt \, dt \overset{\text{sgn lichá, } \cos(nt) \text{ sudá}}{=} 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} t \sin nt \, dt \overset{\text{sgn lichá, } \sin nt \text{ lichá} \rightsquigarrow \text{dohromady sudá}}{=} \frac{2}{\pi}\int_0^{\pi} \underbrace{\operatorname{sgn} t}_{=1} \sin nt \, dt = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{(-1)^{n+1}}{n} + \frac{1}{n}\right)$$

$$= \begin{cases} 0, & n \text{ sudé} \\ \dfrac{4}{\pi n}, & n \text{ liché} \end{cases}$$

$\Rightarrow$ Fourierova řada je tedy $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}\left((-1)^{n+1} + 1\right)\sin nx = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin[(2k-1)x]$

Zatím nevíme, zda její součet je f.

Konvergence: $f \in BV([-\pi,\pi])$, protože je monotonní na $[-\pi,\pi]$.

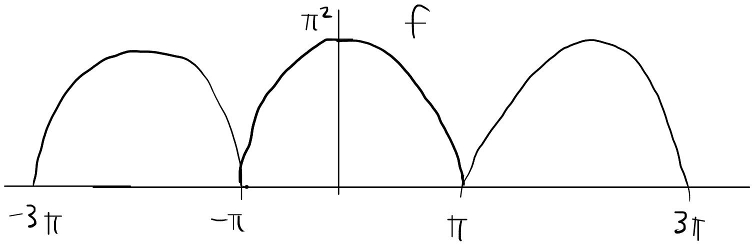
Jordan–Dirichlet $\Rightarrow$ Fourierova řada konverguje bodově k $\tilde{f}$

$$\tilde{f}(x) = \begin{cases} 1, & x \in (2\ell\pi, (2\ell+1)\pi) \\ 0, & x = \ell\pi \\ -1, & x \in ((2\ell-1)\pi, 2\ell\pi). \end{cases}$$

Tzn.
$$\tilde{f}(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin[(2k-1)x], \quad x \in \mathbb{R}.$$

Konvergence není stejnoměrná, protože $\tilde{f}$ není spojitá.

$\boxed{2}$ $f(x) = \pi^2 - x^2$, $x \in (-\pi, \pi)$, navíc $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = ?$



Dodefinujeme na $\mathbb{R}$: $f(x) = \pi^2 - (x - 2\pi l)^2$, $x \in [(2l-1)\pi, (2l+1)\pi)$.

Pak f je spojitá na $\mathbb{R}$.

Spočteme Fourierovy koeficienty:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \pi^2 - x^2\,dx \overset{\text{sudost } f}{=} \frac{2}{\pi}\int_0^{\pi} \pi^2 - x^2\,dx = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) = \pi^2\, 2\left(1 - \frac{1}{3}\right) = \pi^2 \cdot \frac{4}{3}$$

$n \geq 1$: $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos nx\,dx \overset{\text{sudost}}{=} \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\left(\int_0^{\pi} \pi^2 \cos nx\,dx - \int_0^{\pi} x^2 \cos nx\,dx\right) =$$

$$= \frac{2}{\pi}\underbrace{\left[\pi^2 \frac{\sin nx}{n}\right]_0^{\pi}}_{0} - \frac{2}{\pi}\left(\underbrace{\left[x^2 \frac{\sin nx}{n}\right]_0^{\pi}}_{0} - \int_0^{\pi} 2x \frac{\sin nx}{n}\,dx\right) = \frac{4}{\pi n}\int_0^{\pi} x \sin nx\,dx =$$

$$= \frac{4}{\pi n}\left(\left[x \cdot \left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n}\,dx\right) = \frac{4}{\pi n}\left(\frac{\pi \cdot (-1)^{n+1}}{n} + \underbrace{\left[\frac{\sin nx}{n^2}\right]_0^{\pi}}_{0}\right) = \frac{4 \cdot (-1)^{n+1}}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\sin nx\,dx \overset{f \text{ sudá, } \sin nx \text{ lichá}}{=} 0$$

$\Rightarrow$ Fourierova řada má tvar $\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos nx$

Konvergence: • f je spojitá na $\mathbb{R}$

• $f \in BV([-\pi, \pi])$, $\left[f \in C^1_{\text{na } [-\pi,\pi]} \Rightarrow f \in Lip([-\pi,\pi]) \Rightarrow f \in BV([-\pi,\pi]) \right]$

↳ i na každém omezeném intervalu $[a, b] \subset \mathbb{R}$. $\forall\, a<b$. $f \in BV([a,b])$

Jordan – Dirichlet na $(-2\pi, 2\pi)$ $\Rightarrow$ $f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2}\cos nx$, $x \in \mathbb{R}$,

a navíc konvergence je stejnoměrná na $[-\pi, \pi]$

$\Rightarrow$ je stejnoměrná na $\mathbb{R}$ díky 2π-periodicitě.

Součty řad: dosadíme $x = \pi$: $0 = \pi^2 - \pi^2 = f(\pi) = \frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n^2}(-1)^n}_{=-1}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

dosadíme $x = 0$: $\pi^2 = f(0) = \frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cdot 1$

$$\Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Pozorování: $f:(-\pi,\pi)\to\mathbb{R}$ $\quad$ f lichá $\Rightarrow a_n=0$

f sudá $\Rightarrow b_n=0$

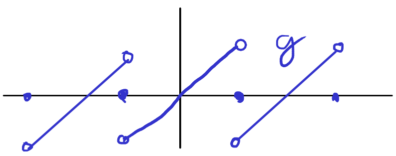
$\boxed{3}$ $f(x)=x$, $\quad x\in(0,\pi)$. $\quad$ Fourierovu řadu $\quad$ (a) $\sum_{n=1}^{\infty} b_n \sin nx$

(b) $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$

Funkci máme dodefinovat 2π-periodicky, máme tedy volnost v tom jak definovat f na $(-\pi,0]$.

(a) Chceme sinovou řadu, tj. chceme $a_n=0$. Dodefinujme proto f na lichou fci g na $(-\pi,\pi)$: $g(x)=x$, $x\in(-\pi,\pi)$

Dodefinujme g 2π-periodicky: $g(x)=\begin{cases} x-2\pi l & , x\in((2l-1)\pi,(2l+1)\pi) \\ 0 & , x=(2l+1)\pi\end{cases}$



Fourierovy koeficienty: $a_n=0$ $\quad$, díky lichosti:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx=\frac{2}{\pi}\left(\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi}-\int_0^{\pi}1\left(\frac{-\cos nx}{n}\right)dx\right)=$$

$$=\frac{2}{\pi}\pi\frac{(-1)^{n+1}}{n}-\underbrace{\frac{2}{\pi}\left[\frac{-\sin nx}{n^2}\right]_0^{\pi}}_{0}=\frac{2(-1)^{n+1}}{n}$$

Four. řada je $\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$.

jednostranné limity v $(2l+1)\pi$ sedí

$g\in BV([-\pi,\pi])$ $\overset{\text{Jordan-Dirichlet}}{\rightsquigarrow}$ $g(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$, $\quad x\in\mathbb{R}$

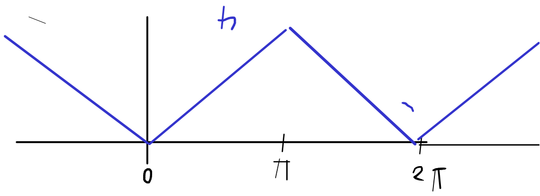
Tedy máme $f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$, $\quad x\in(0,\pi)$

Dosazením $\underline{x=1}$: $\quad 1=f(1)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n=\frac{1}{2}$.

(b) dodefinujte f na $(-\pi,\pi)$ jako sudou funkci:

$h(x)=|x|$, $x\in(-\pi,\pi)$. dodefinujeme 2π-periodicky:

$h(x)=|x-2\pi l|$, $\quad x\in((2l-1)\pi,(2l+1)\pi]$



Pak h je spojitá na $\mathbb{R}$.

Navíc je omezené variace na každém omezeném intervalu, protože je po částech C^1.

Jordan-Dirichlet $\Rightarrow$ Fourierova řada konverguje stejnoměrně k h.

Výpočet Fourierovy řady: $b_n = 0$ (h sudá) $\qquad a_0 \overset{h\text{ sudá}}{=} \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \frac{2}{\pi}\,\frac{\pi^2}{2} = \pi$

$n\geq 1$: $a_n \underset{h\text{ sudá}}{=} \frac{2}{\pi}\int_0^\pi h(x)\cos nx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi}\left(\underbrace{\left[x\frac{\sin nx}{n}\right]_0^\pi}_{0} - \int_0^\pi 1\cdot\frac{\sin nx}{n}\,dx\right) =$

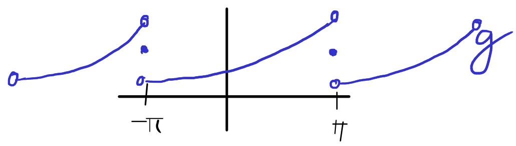
$$= -\frac{2}{\pi}\left[\frac{-\cos nx}{n^2}\right]_0^\pi = \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ sudé} \\ -\dfrac{4}{\pi n^2}, & n \text{ liché.}\end{cases}$$

$$\Rightarrow h(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos[(2k+1)x], \quad x\in\mathbb{R}$$

Dosazení $x=0$: $\quad 0 = h(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos 0 \Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$

$\boxed{4}$ $\quad f(x) = e^x, \quad x\in(-\pi,\pi)$

Dodefinujeme na $\mathbb{R}$ jako



$$g(x) = \begin{cases} e^{x-2\pi\ell}, & x\in((2\ell-1)\pi,(2\ell+1)\pi) \\ \dfrac{e^{\pi}+e^{-\pi}}{2}, & x = (2\ell+1)\pi\,. \end{cases}$$

Pak g je konečná variace na omez. intervalech, a v bodech nespojitosti platí $g(x) = \dfrac{\lim_{t\to x^+} g(t) + \lim_{t\to x^-} g(t)}{2}$. Jordan – Dirichlet $\Rightarrow$ Fourierova řada konverguje bodově na $\mathbb{R}$ ke g.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$

$n\geq 1$: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx = \frac{1}{\pi}\left(\underbrace{\left[e^x\frac{\sin nx}{n}\right]_{-\pi}^{\pi}}_{=0} - \underbrace{\int_{-\pi}^{\pi} e^x\frac{\sin nx}{n}\,dx}_{=\frac{\pi}{n}b_n}\right) = -\frac{1}{\pi n}\left(\left[e^x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi}\right.$

$\left. - \underbrace{\int_{-\pi}^{\pi} e^x\frac{(-\cos nx)}{n}\,dx}_{\frac{\pi a_n}{n}}\right) = -\frac{(-1)^n}{\pi n^2}\left(-e^{\pi} + e^{-\pi} + \pi a_n\right)$

$-\frac{1}{n}b_n = a_n$

$\Rightarrow \quad \underbrace{\left(1 + \frac{1}{n^2}\right)}_{=\frac{n^2+1}{n^2}} a_n = \frac{1}{\pi n^2}\left(e^{\pi} - e^{-\pi}\right) \quad \Rightarrow a_n = \frac{(-1)^n}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^{n+1}n}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$

$$g(x) = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n\left(e^{\pi}-e^{-\pi}\right)}{\pi(n^2+1)}\left(\cos nx - n\sin nx\right), \quad x\in\mathbb{R}.$$

Dosadíme $x=0$: $\quad 1 = e^0 = g(0) = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n\left(e^{\pi}-e^{-\pi}\right)}{\pi(n^2+1)}\cdot 1$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi}{e^{\pi}-e^{-\pi}} - \frac{1}{2} = \frac{\pi}{2\sinh\pi} - \frac{1}{2}, \quad \left[\sinh x = \frac{e^x - e^{-x}}{2}\right]$$

$\boxed{5}$ (a) $f(x) = \sin kx$, $x \in (\pi, \pi]$ (b) $f(x) = \cos kx$, $x \in (-\pi, \pi]$ $(k \in \mathbb{N})$

(a) $n \geq 0$: $a_n = 0$, f lichá

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \sin nx \, dx = \begin{cases} \xrightarrow{k=n} \frac{1}{\pi}\int_{-\pi}^{\pi} (\sin kx)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2kx}{2} dx = \frac{1}{\pi}\left(\left[\frac{x}{2}\right]_{-\pi}^{\pi} - \underbrace{\left[\frac{\sin 2kx}{2k}\right]_{-\pi}^{\pi}}_{0}\right) = \frac{1}{\pi}\cdot\frac{1}{2}(\pi-(-\pi)) = 1 \\ \xrightarrow{k \neq n} \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} \sin kx \sin nx \, dx}_{=: I_n} = \frac{1}{\pi}\left(\underbrace{\left[\frac{-\cos kx}{k}\sin nx\right]_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} n \cos nx \, dx\right) = \end{cases}$$

$$= \frac{1}{\pi}\frac{n}{k}\left(\left[\frac{\sin kx}{k}\cos nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin kx}{k}(-n \sin nx) dx\right) = \frac{1}{\pi}\frac{n^2}{k^2}\underbrace{\int_{-\pi}^{\pi} \sin kx \sin nx \, dx}_{I_n}$$

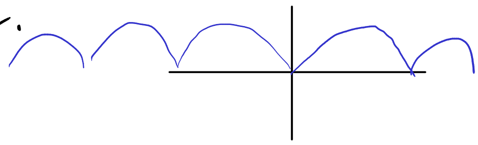
$$\Rightarrow \left(1 - \frac{n^2}{k^2}\right) I_n = 0 \Rightarrow I_n = 0, \text{ tzn. } b_n = 0 \text{ pro } n \neq k$$

tedy Fourierova řada f je opravdu $\sin kx$.

(b) analogicky

$\boxed{6}$ $f(x) = \sin 3x + 4x$, $x \in (-\pi, \pi]$. Zkombinujte $\boxed{3}$ a $\boxed{5}$

$\boxed{7}$ $f(x) = \sin x$, $x \in (0, \pi)$, pomocí kosinové řady.



Dodefinujeme f jako $g(x) = |\sin x|$, $x \in \mathbb{R}$

Pak g je spojitá a konečné variace na omez. intervalech (je po částech C^1), tedy Jordan–Dirichlet $\Rightarrow$ Fourierova řada konverguje ke g na $\mathbb{R}$ (stejnoměrně).

$$a_0 \overset{\text{sudost}}{=} \frac{2}{\pi}\int_0^{\pi} \sin x = \frac{2}{\pi}[\cos x]_0^{\pi} = \frac{4}{\pi}$$

$\underline{n \geq 1}$: $$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx = \frac{2}{\pi}\left(\underbrace{\left[\sin x \frac{\sin nx}{n}\right]_0^{\pi}}_{0} - \int_0^{\pi} \cos x \frac{\sin nx}{n} dx\right) =$$

$$= \frac{-2}{\pi n}\left(\left[\cos x \frac{(-\cos nx)}{n}\right]_0^{\pi} - \int_0^{\pi} \sin x \frac{\cos nx}{n}\right) = \frac{-2}{\pi n}\left(\frac{(-1)^n}{n} + \frac{1}{n}\right) + \frac{1}{n^2}\underbrace{\frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx}_{a_n}$$

$$\Rightarrow \frac{1-n^2}{\cancel{n^2}} a_n = \left(1 - \frac{1}{n^2}\right) a_n = \frac{-2}{\pi \cancel{n^2}}\left((-1)^n + 1\right) \Rightarrow a_n = \frac{-2}{\pi(n^2-1)}\left((-1)^n + 1\right) =$$

$$= \begin{cases} \frac{-4}{\pi(n^2-1)}, & n \text{ sudé} \\ 0 & n \text{ liché} \end{cases}$$

$$\Rightarrow g(x) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{-4}{\pi(4k^2-1)} \cos 2kx$$