**Deadline**: 6.10.2021, 11:30

# Laplace equation on $\mathbb{R}^d$

We will consider the Laplace equation in the whole space

$$\Delta u = 0 \quad \text{in } \mathbb{R}^d.$$

Recall that the definition of the *fundamental solution of the Laplace equation* is

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{if } d = 2, \\ \dfrac{1}{d(d-2)\alpha_d}\dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3, \end{cases} \qquad x \in \mathbb{R}^d \setminus \{0\},$$

where $\alpha_d$ is the volume of the unit ball in $\mathbb{R}^d$.

Recall also the definition of the space of *distributions* $\mathcal{D}'(\mathbb{R}^d)$:
We say that $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ if it is a linear functional $\Lambda\colon \mathcal{C}_c^\infty(\mathbb{R}^d) \to \mathbb{R}$, continuous with respect to some topology[1]. The topology is not so important for us, we only need to know that any function $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ can be understood as a distribution $\lambda_g \in \mathcal{D}'(\mathbb{R}^d)$, defined as

$$\lambda_g(\varphi) = \int_{\mathbb{R}^d} g(x)\varphi(x)\,\mathrm{d}x, \quad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d),$$

and that $\delta_0$ defined below is also a distribution.
For a distribution $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ we define its *(i-th partial) distributional derivative*[2] $\dfrac{\partial \Lambda}{\partial x_i} \in \mathcal{D}'(\mathbb{R}^d)$ as (note the minus sign)

$$\frac{\partial \Lambda}{\partial x_i}(\varphi) = -\Lambda\left(\frac{\partial \varphi}{\partial x_i}\right), \quad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

For higher derivatives we proceed inductively, i.e. $\dfrac{\partial^2 \Lambda}{\partial x_j \partial x_i}(\varphi) = \Lambda\left(\dfrac{\partial^2 \varphi}{\partial x_j \partial x_i}\right)$ etc.

**1.** Show that $\Phi$ defined above deserves the name “fundamental solution”. That is, prove that $\Phi$ satisfies the equation

$$-\Delta \Phi = \delta_0$$

in the sense of distributions on $\mathbb{R}^d$, where $\delta_0 \in \mathcal{D}'(\mathbb{R}^d)$ is the Dirac distribution at 0, that is $\delta_0(\varphi) = \varphi(0)$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. (Here “$\Phi$” is understood as the distribution $\Lambda_\Phi$ and “$\Delta$” is the corresponding distributional derivative.)

**2.** Further, let[3] $f \in \mathcal{C}_c^2(\mathbb{R}^d)$. Prove that the function $u = \Phi * f$, that is,

$$u(x) = \Phi * f(x) = \int_{\mathbb{R}^d} \Phi(y) f(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d$$

is a solution to the nonhomogeneous problem (also called the *Poisson equation*)

$$-\Delta u = f \quad \text{in } \mathbb{R}^d$$

in the classical sense.

[1] This topology on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is given, for instance, by the metric $\rho(\varphi,\psi) = \sum_{N=0}^{\infty} \frac{1}{2^N} \min\{\|\varphi - \psi\|_{\mathcal{C}^N(\mathbb{R}^d)}, 1\}$.

[2] Sometimes this is denoted as $D_i\Lambda$ to emphasize the fact that this is not the classical derivative.

[3] The condition on $f$ is rather restrictive and can be relaxed in various ways. We assume $\mathcal{C}_c^2$ for simplicity.

**Hints**

Remember that two distributions are equal if they attain the same value on every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

The essential step of the proof is to isolate the singularity of $\Phi$ at 0. That is, split the integral to the ball $B(0,\varepsilon)$ and $\mathbb{R}^d \setminus B(0,\varepsilon)$. Then you can integrate by parts (twice) in the second one, where $\Phi$ behaves nicely. Finally, pass with $\varepsilon \to 0$.

(You should have already seen this in the course on the classical theory of PDEs.)