Deadline: 15.12.2021, 11:30

Regularity II

Let us consider the following mixed boundary value problem for given $f \in L^2(\Omega)$, (here n denotes the outer unit normal to $\partial\Omega$):

$$\begin{aligned} -\Delta u = f \text{ in } \Omega = B(0,1) \cap \mathbb{R}^2_+ &= \{(x,y) \in \mathbb{R}^2 : x, y > 0, x^2 + y^2 < 1\}, \\ \frac{\partial u}{\partial n} := \nabla u \cdot n = 0 &\text{ on } (0,1) \times \{0\} \cup \{0\} \times (0,1), \\ u = 0 &\text{ on } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1,\, x, y > 0\}. \end{aligned}$$

1. Formulate the problem weakly and show the existence of a unique weak solution u.

2. We extend u to be defined on the unit disk $D := B(0,1)$ by even reflection, so that

$$u(x,y) = u(-x,y) = u(x,-y) = u(-x,-y), \quad (x,y) \in \Omega.$$

Show that u solves the reflected equation (formulate it) on D and moreover $u \in W^{2,2}_{\mathrm{loc}}(D)$.

3. Suppose that we have the same problem but with coefficients $A\colon \Omega \to \mathbb{R}^{2\times 2}$:

$$\begin{aligned} -\operatorname{div}(A\nabla u) &= f \text{ in } \Omega, \\ A\nabla u \cdot n &= 0 \text{ on } (0,1) \times \{0\} \cup \{0\} \times (0,1), \\ u &= 0 \text{ on } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1,\, x, y > 0\}. \end{aligned}$$

Under what assumptions on A can we reach the same conclusions as in problems 1, 2?