Deadline: 5.1.2022, 11:30

Linear parabolic equations II

1. (Energy decay of the heat equation)
Let Ω be a Lipschitz domain and $u_0 \in L^2(\Omega)$. Let u be a weak solution[1] to the problem

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,\infty)\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ u &= 0 && \text{on } (0,\infty)\times\partial\Omega. \end{aligned}$$

Show that there exists a constant $\lambda_1 > 0$ such that

$$\|u(t,\cdot)\|_{L^2(\Omega)} \le \mathrm{e}^{-\lambda_1 t}\,\|u_0\|_{L^2(\Omega)}$$

for $t \in (0,\infty)$. Show that, in fact, the constant λ_1 is the first eigenvalue of the Laplace operator with respect to the homogeneous Dirichlet boundary conditions. That is λ_1 the smallest λ such that there exists a nonzero weak solution to

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

[1] in the sense as defined in the lecture