**Deadline**: 27.10.2021, 11:30

# Dual of $L^p$spaces

We will prove that $L^p(\Omega)^* = L^{p'}(\Omega)$ for $p \in (1, \infty)$:

Let $\Omega \subset \mathbb{R}^n$ be measurable and $p \in (1, \infty)$.

1. For $f \in L^{p'}(\Omega)$ define the functional $T_f \colon L^p(\Omega) \to \mathbb{R}$ by[1] $\langle T_f, g\rangle = \int_\Omega fg$, $g \in L^p(\Omega)$. Show that $T_f \in (L^p(\Omega))^*$ and $\|T_f\|_{(L^p(\Omega))^*} = \|f\|_{L^{p'}(\Omega)}$.
2. Let $T \in (L^p(\Omega))^*$. Show that there exists a solution to the minimization problem

$$\min_{g \in L^p(\Omega)} \int_\Omega \frac{|g|^p}{p} - \langle T, g\rangle.$$

   *Hint: Use the direct method as described in the lecture.*

3. *(Optional, as the lecture has not yet discussed the Euler–Lagrange equation)*
   Using the above, show that for $T \in (L^p(\Omega))^*$ there exists $f_T \in L^{p'}(\Omega)$, such that

$$\forall g \in L^p(\Omega) : \langle T, g\rangle = \int_\Omega f_T g.$$

[1]The notation $\langle T_f, g\rangle$ stands for $T_f(g)$.