Deadline: 3.11.2021, 11:30

Euler–Lagrange equation

1. *(continuation from last week)*
Let $\Omega \subset \mathbb{R}^n$ be measurable and $p \in (1, \infty)$.
For a given $T \in (L^p(\Omega))^*$, define the functional $\mathcal{F}\colon L^p(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(g) = \int_\Omega \frac{|g|^p}{p} - \langle T, g \rangle, \quad g \in L^p(\Omega).$$

(Recall that last week we proved that $\mathcal{F}$ has a minimizer.)

- Write the Euler–Lagrange equation for the functional $\mathcal{F}$.
- Using this, prove that there exists $f_T \in L^{p'}(\Omega)$, such that

$$\forall g \in L^p(\Omega) : \langle T, g \rangle = \int_\Omega f_T g.$$

- Conclude that $(L^p(\Omega))^*$ is isomorphic to $L^{p'}(\Omega)$ (using also the results from last week).

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\varphi \in \mathcal{C}^1(\mathbb{R}^n)$ be such that $|\nabla \varphi|$ is bounded on $\mathbb{R}^n$. Define the functional $\mathcal{J}\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}(u) = \int_\Omega \varphi(\nabla u)\,\mathrm{d}x, \quad u \in W_0^{1,2}(\Omega).$$

- Compute the first variation[1] of $\mathcal{J}$ at $u_0 \in W_0^{1,2}(\Omega)$. (Don't forget to prove that it exists.)
- Write the Euler–Lagrange equation for $\mathcal{J}$.

[1] Also called the Gateaux derivative.