Poincaré inequalities

In the case of second derivative, the following Poincaré-type inequality holds (you don't need to prove it):

Theorem. Let $\Omega \subset \mathbb{R}^n$ be Lipschitz, $p \in [1, \infty)$ and let $\Gamma \subset \partial\Omega$ be a set with positive $(n-1)$–dimensional measure such that Γ **is not contained in a hyperplane**[1]. Then there is a constant $C > 0$ such that

$$\forall u \in W^{2,p}(\Omega) : \|u\|_{W^{2,p}(\Omega)} \leq C \left(\left\|\nabla^2 u\right\|_{L^p(\Omega)}^p + \int_\Gamma |u|^p \, \mathrm{d}S \right)^{\frac{1}{p}}.$$

1. Show by counterexample that the assumption **in bold** cannot be dropped in the above theorem.

2. Consider the functions u_δ, $0 < \delta < 1$ defined on the unit ball $B = B(0,1) \subset \mathbb{R}^n$ for $n \geq 2$ by

$$u_\delta(x) = \begin{cases} \dfrac{|x|}{\delta}, & \text{if } |x| < \delta \\ 1, & \text{if } \delta \leq |x| < 1. \end{cases} \tag{1}$$

Verify that $u_\delta \in W^{1,1}(B)$ and show that $u \to 1$ in $W^{1,1}(B)$.
Using this, show that "being zero at one point is not enough for Poincaré inequality to hold": that is, there is no $C > 0$ such that for every $u \in W^{1,1}(B)$ with $u(0) = 0$, where 0 is a Lebesgue point[2] of u it holds

$$\|u\|_{L^1(B)} \leq C\|\nabla u\|_{L^1(B)}. \tag{2}$$

[1] A *hyperplane* is an $(n-1)$–dimensional affine subspace of $\mathbb{R}^n$.
[2] As u is defined only almost everywhere, this is one way to make sense of "$u(0) = 0$".