**Deadline**: 8.12.2021, 11:30

# Regularity

**1.** For a function $u\colon \Omega \to \mathbb{R}$ defined on some open $\Omega \subset \mathbb{R}^n$ denote its *difference quotients* (for given $\delta > 0$, $0 \neq h \in \mathbb{R}$) by[1]

$$D_h^i u(x) = \frac{u(x + he_i) - u(x)}{h}, \quad x \in \Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}, |h| < \delta.$$

Find $u \in L^1(\Omega)$ satisfying $u \notin W^{1,1}_{\mathrm{loc}}(\Omega)$ such that there is $C > 0$ with

$$\forall i = 1, \ldots, n,\ 0 \neq h \in \mathbb{R}, \delta > |h| : \|u_{h,i}\|_{L^1(\Omega_\delta)} \leq C.$$

**Remark.** This example shows that the theorem saying "bounded difference quotients in $L^p$ imply weak derivative in $L^p$", which is used in the proof of regularity for linear elliptic problems, does not hold for $p = 1$.

**2.** (Harnack's inequality using the mean value formula)
Let $\Omega$ be open and $K \subset \Omega$ compact. Show that there extists a $C > 0$ such that every positive harmonic function, that is $u\colon \Omega \to \mathbb{R}$ satisfying $u > 0$ and $\Delta u = 0$ in $\Omega$, satisfies the *Harnack's inequality*

$$\sup_{x \in K} u(x) \leq C \inf_{x \in K} u(x).$$

You may use without proof that $u$ satisfies the following *mean value formula*

$$\forall B(x, r) \subset \Omega : u(x) = \frac{1}{|B(x, r)|} \int_{B(x,r)} u(y) \, \mathrm{d}y$$

[1] The reason we are considering the domain $\Omega_\delta$ instead of $\Omega$ is merely technical – we need $u(x + he_i)$ to be defined, i.e. $x + he_i \in \Omega$.