Nonlinear Differential Equations

- Exam: Oral Assessment
- Webpage + suggested reading:
  https://www.karlin.mff.cuni.cz/~congreve/teaching.php?c=SS2023_NDE

- Practical:
  - Exercise posted on webpage every week
  - Discuss solutions during following practical
  - Credit for active participation in practicals

Topics

1) Introduction & Basic Concepts/Theorems
   - Basics from functional analysis
   - Fixed point theorems
2) Monotone & Potential operators
   - Theory
   - Existence of solutions
3) Nonlinear Differential Equations
   - Divergence form
   - Carathéodory's growth conditions
   - Nemycky operators
   - Application of monotone/potential operator theory
   - Existence of solutions
4) Numerical solutions.

1. Introduction

Motivation – practical

1.1 Definitions & Basic Theorems

Definition 1.1 Let X & Y be normed linear spaces & A be a general nonlinear operator defined with domain $\mathcal{D}(A) \subseteq X$ and range $\mathcal{R}(A) \subseteq Y$.

1) A subset $K \subset X$ is called (sequentially) compact in X if it is closed and every sequence $\{u_n\} \subset K$ has a convergent subsequence in X.
   Equivalently: A subset $K \subset X$ is compact if for every open covering a finite subcover can be selected
2) A subset $K \subset X$ is called precompact (or relatively compact) if its closure is compact in X.
3) $A: X \to Y$ is a compact operator if the image of the bounded subset $K \subset \mathcal{D}(A)$ under A is a precompact subset of Y; i.e. $\overline{A(K)}$ is compact.
   Equivalently: $A: X \to Y$ is compact on the set $\mathcal{D}(A)$ if for any bounded sequence $\{u_n\} \subset \mathcal{D}(A)$ the sequence $\{Au_n\}$ contains a convergent subseq. in Y.

Note: If $A: X \to Y$ is compact & linear then it is continuous.

Notations: $\|u\|$ – Norm of $u \in X$ in normed space X

$\mathcal{L}(X,Y)$ – Set continuous linear operators from X to Y

$C_0(X,Y)$ – Set of linear compact

Let X be normed linear space; then, set of all linear & continuous functionals on X form vector space X^*, where $\langle \cdot , \cdot \rangle$ is a dual relationship; i.e, $\langle x^*, z \rangle$ is value of $x^* \in X^*$ at point $z \in X$ (or (z, x^*), or $x^*(z)$).

Recall — dual space is continuous. Space of all continuous functionals over X^* (normed bidual space) is X^{**}.

<u>Definition 1.2</u> The Banach space X is called reflexive if the canonical map $J: X \to X^{**}$ defined by

$$\langle Jx, x^* \rangle = \langle x^*, x \rangle \quad \forall x^* \in X^*, x \in X$$

is surjective (J embeds X into X^{**}).

<u>Note</u> In reflexive Banach space the canonical map J is linear, isomorphic, and an isometric isomorphism

X reflexive $\Rightarrow$ dual X^* reflexive

X^* reflexive & complete $\Rightarrow$ X reflexive

<u>Definition 1.3</u> Let X be a normed linear space. Then, $\{u_n\} \in X$ converges weakly (X^*-weakly, seq. $\{u_n\}$ w-converges) to the point $u \in X$ if for every $x^* \in X^*$ it holds that $\langle x^*, u_n \rangle \to \langle x^*, u \rangle$. We denote weak convergence as $u_n \rightharpoonup u$.

The sequence $\{u_n^*\} \subset X^*$ w^*-converges (X-weakly) to $x^* \in X^*$ for each $x \in X$ if $\langle x_n^*, x \rangle \to \langle x^*, x \rangle$. We write weak convergence of these functionals as $x_n^* \xrightarrow{w^*} x^*$.

<u>Note</u> If X is reflexive then w- and w^*-convergence coincide, so instead of w^*-convergence we use $x_n^* \rightharpoonup x_n$.

Theorem 1.4 The normed space X is reflexive if and only if each sequence in X has a convergent subsequence in X.

Proposition 1.5

1. The sequence $\{u_n\} \subset X$ converges weakly (X^*-weakly) to $u \in X$ if the sequence $\{u_n\}$ is bounded and $\lim_n \langle x^*, u_n \rangle = \langle x^*, u \rangle$ for each continuous linear functional x^* of some dense subset M of X^*.
2. If the sequence $\{u_n\}$ in a Banach space X weakly converges to the point u, then $\|u\| \leq \liminf \|u_n\|$
3. Convex, closed subset K of a Banach space X is weakly closed
$$\{u_n\} \subset K, \ u_n \rightharpoonup u \Rightarrow u \in K$$
4. In a reflexive Banach space, every closed ball is weakly compact; i.e, each sequence of elements from this ball contains a weakly convergent subsequence with a limit in this ball (Every bounded sequence contains a weakly convergent subsequence).
5. Let $\{u_n\}$ be a bounded sequence in the reflexive Banach space X and let all weakly convergent subsequences have u as their (weak) limit; Then, the whole sequence converges to u.
6. If a closed ball in a reflexive Banach space lies in the union of a family of weakly open sets; Then, the ball lies in the union of finite subcollection of these sets.

7. Let K be a convex, bounded, and closed set in a reflexive Banach space X; then this set is weakly compact.

8. Let K be a non-empty, closed, and convex set in a Hilbert space H with norm $\|\circ\|$ and inner product $(\circ,\circ)$. Then, for every $x \in H$ there exists a $u \in K$, such that $\|x-u\| = \min\limits_{v \in K} \|x-v\|$

This element can be characterise as

$$u \in K \quad (x-u, v-u) \leq 0 \quad \text{for all } v \in K.$$

Moreover, define $u = P_K x$, then

$$\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\| \quad \forall x_1, x_2 \in H$$

(P_K is projection onto K).

If K is a closed linear subspace in H; then P_K is a continuous linear operator. The element u can be charactised by $u \in K \quad (x-u, v-u) = 0 \quad \forall v \in K.$

9. Let K be a convex closed subset of a Banach space X. Then, for every $x \notin K$ there exists a functional $x^* \in X^*$ such that

$$\langle x^*, x \rangle > \sup_{y \in K} \langle x^*, y \rangle$$

10. Uniform boundness principle

Let $G \subset \mathcal{L}(X,Y)$, where X is a Banach space & Y is a normed linear space. Then, the following are equivalent:

a) $\sup\{\|A\| : A \in G\} < +\infty$

b) $\sup\{\|Ax\| : A \in G\} < +\infty$ for every $x \in X$.

Definition 1.6 Let X be a topological space. Then, the function φ is called lower semi-continuous on X, if for any $x_0 \in X$ it holds that
$$\varphi(x_0) \leq \liminf_{x \to x_0} \varphi(x)$$
and upper semi-continuous on X, if for any $x_0 \in X$ it holds
$$\varphi(x_0) \geq \limsup_{x \to x_0} \varphi(x).$$

Definition 1.7 Let X be a normed linear vector space and $U \subseteq X$ an open set. Then, the real function φ is called lower semi-continuous on U, if for any $x_0 \in U$ and every sequence $\{x_n\} \subset U$ which strongly converges to x_0 (in the norm) it holds that
$$\varphi(x_0) \leq \liminf_{n \to \infty} \varphi(x_n)$$
and, similarly, upper semi-continuous if
$$\varphi(x_0) \geq \limsup_{n \to \infty} \varphi(x_n)$$

Definition 1.8 Let X be a normed linear vector space. Then, the real function φ is called weakly lower semi-continuous on X, if for any $x_0 \in X$ and every weakly convergent sequence $\{x_n\}$ with limit x_0 it holds that
$$\varphi(x_0) \leq \liminf_{n \to \infty} \varphi(x_n)$$
and, similarly, weakly upper semi-continuous if
$$\varphi(x_0) \geq \limsup_{n \to \infty} \varphi(x_n)$$

Remark Weakly lower semi-continuous $\Rightarrow$ lower semi-cont.

1.2 Fixed Point Theorems

Definition 1.9 Let (X,d) be a metric space (with metric $d(\cdot,\cdot)$), and $T: X \to X$ a mapping; then, $\bar{x} \in X$ is called a fixed point of T if $T\bar{x} = \bar{x}$.

Remark Typically X will be a normed vector space, Hilbert space, or Banach space, with d being the norm of the difference: $d(u,v) = \|u-v\|$

Definition 1.10 Let (X,d) be a metric space and $M \subset X$. Then, a mapping $T: M \to X$ is called

a) contractive if $d(T(x), T(y)) < d(x,y)$, $x \neq y$, $x,y \in M$

b) strongly contractive if $\exists k \in (0,1)$ such that

$$d(T(x), T(y)) \leq k\, d(x,y) \quad \forall x,y \in M.$$

Theorem 1.11 (Banach fixed point theorem/Contraction Mapping Th.)

Let (X,d) be a complete metric space, $M \subseteq X$ a non-empty closed subset, and $T: M \to M$ be strongly contractive. Then,

a) T has a unique fixed point $\bar{x} \in M$

b) the fixed point iteration

$$x_{n+1} = T(x_n), \quad n \geq 0$$

converges to $\bar{x}$ (as $n \to \infty$) for any starting $x_0 \in M$.

c) $d(\bar{x}, x_n) \leq \frac{k^n}{1-k} d(x_0, x_1)$ (a priori)

$d(\bar{x}, x_n) \leq \frac{k}{1-k} d(x_n, x_{n-1})$ (a posteriori)

Proof

1. Convergence:

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n))$$
$$\leq k\, d(x_{n-1}, x_n)$$
$$\leq k^n d(x_1, x_0) \qquad \text{(recursive app.)}$$

By triangle inequality, $m \geq 1$,

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+m-1}, x_{n+m})$$
$$\leq k^n d(x_1, x_0) + k^{n+1} d(x_1, x_0) + \dots + k^{n+m-1} d(x_1, x_0)$$
$$= k^n d(x_1, x_0) \underbrace{\sum_{j=0}^{k-1} k^j}_{= \frac{1-k^m}{1-k}}$$
$$\leq \frac{k^n}{1-k} d(x_0, x_1)$$

For $n, m \to \infty$ $(n > m)$ we have that $d(x_n, x_m) \to 0$

$\Rightarrow \{x_n\}$ is a Cauchy sequence with limit $\bar{x} \in X$

M is closed & $\{x_n\} \subset M \Rightarrow \bar{x} \in M$

2. $\bar{x}$ is a fixed point of T: Contraction $\Rightarrow$ T continuous

$$\Rightarrow \bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\bar{x})$$

3. Uniqueness. Suppose $\bar{x}_1 \neq \bar{x}_2$ are two fixed points in M; then,

$$d(\bar{x}_1, \bar{x}_2) = d(T(\bar{x}_1), T(\bar{x}_2))$$
$$\leq k\, d(\bar{x}_1, \bar{x}_2)$$

which, as $k < 1$ and $d(\bar{x}_1, \bar{x}_2) \neq 0$, is a contradiction $\square$