Theorem 1.12 (Brower's FP Theorem)

Let $\overline{B_1(0)} = \{x \in \mathbb{R}^n : \|x\| \leq 1\} \subset \mathbb{R}^n$ (unit ball) and $f : \overline{B_1(0)} \to \overline{B_1(0)}$ be continuous. Then f has a fixed point in $\overline{B_1(0)}$.

Proof

- We need a utility lemma (Retraction principle):
  There is no C^1-mapping $F : \overline{B_1(0)} \to \partial B_1(0)$ with $f(x) = x$ for all $x \in \partial B_1(0)$

- Let $f : \overline{B_1(0)} \to \overline{B_1(0)}$ be continuous – By Weierstrass approximation theorem there exists sequence of polynomials

$$p_\ell : \overline{B_1(0)} \to \mathbb{R}^n, \quad \ell \geq 1$$

such that

$$\sup_{x \in \overline{B_1(0)}} |f(x) - p_\ell(x)| \leq \frac{1}{\ell}, \quad \ell \geq 1$$

Then, $|p_\ell(x)| \leq |f(x)| + |p_\ell(x) - f(x)| \leq 1 + \frac{1}{\ell}$ $\quad \left(\overset{\text{uniformly}}{x \in \overline{B_1(0)}}\right)$

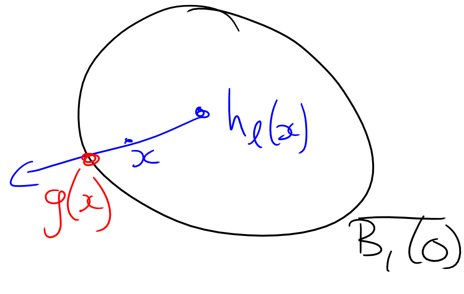
Scaling $p_\ell(x)$ by $\quad h_\ell(x) = \left(1 + \frac{1}{\ell}\right)^{-1} p_\ell(x)$

$$\Rightarrow h_\ell(x) : \overline{B_1(0)} \to \overline{B_1(0)}$$

and $|f(x) - h_\ell(x)| \leq \underbrace{|f(x) - p_\ell(x)|}_{= \frac{1}{\ell} \to 0} + \underbrace{|p_\ell(x) - h_\ell(x)|}_{= \left(1 - \left(1 + \frac{1}{\ell}\right)^{-1}\right)|p_\ell(x)| \to 0}$

$\ell \to \infty$:

- Suppose h_ℓ does <u>not</u> have a fixed point, therefore for any point $x \in \overline{B_1(0)}$, $h_\ell(x) \neq x$. Define $g(x)$ as intersection of line starting at $h_\ell(x)$ and passing through x with the boundary $\partial B_1(0)$



$g: \overline{B_1(0)} \to \partial B_1(0)$ is C^1 and $g(x) = x$ on $\partial B_1(0)$

This contradicts retraction principle lemma

$\Rightarrow h_\ell$ has fixed point $\xi_\ell \in \overline{B_1(0)}$.

$\{\xi_\ell\}_{\ell \geq 1} \subset \overline{B_1(0)}$ is a bounded sequence, and thus, has a convergent subsequence $\xi_{\ell'} \to \bar{\xi} \in \overline{B_1(0)}$.

Then,
$$
\begin{aligned}
|f(\bar{\xi}) - \bar{\xi}| &= \lim_{\ell' \to \infty} |f(\xi_{\ell'}) - \xi_{\ell'}| \\
&\leq \lim_{\ell' \to \infty} \underbrace{|f(\xi_{\ell'}) - h_{\ell'}(\xi_{\ell'})|}_{\to 0} \\
&= 0
\end{aligned}
$$

$\Rightarrow f(\bar{\xi}) = \bar{\xi} \Rightarrow \bar{\xi}$ fixed point of f $\square$

We can now generalise this theorem:

<u>Theorem 1.13</u> Let f be a continuous map defined on a closed, convex, and bounded set $K \subset \mathbb{R}^n$, mapping K to itself; i.e. $f(x) \in K$ for all $x \in K$. Then, there exists a fixed point $\bar{x} \in K$ for f; i.e, $f(\bar{x}) = \bar{x}$.

<u>Proof</u> Let z be interior of K. Then, the set
$$K_1 = \{-z + x, x \in K\} \equiv -z + K$$
which is clearly closed, convex, and bounded. We define the translation T: $Tx = -z + x$, $x \in K$ which maps K to K_1. Function $f_1 y = T \circ f \circ T^{-1} y$ $y \in K$ is continuous and maps K_1 to itself. If y_0 fixed point of f_1 on K_1 then $x_0 = T^{-1} y_0$ is fixed point of f on K. Therefore, sufficient to show f_1 has FP in K_1.

Construct Minkowski functional

$$P_{K_1}(x) = \inf\{\lambda \geq 0, x \in \lambda K_1\}, x \in \mathbb{R}^n$$

where $\lambda K_1 = \{\lambda x, x \in K_1\}$

K_1 is absorbing in $\mathbb{R}^n$, closed, and $0 \in \operatorname{int} K_1$ then

(convex, $\subset \mathbb{R}^n$ & (alg.) int. contains 0) $\quad K_1 = \{x \in \mathbb{R}^n, P_{K_1}(x) \leq 1\}$

If $P_{K_1}(x) < 1 \Rightarrow x \in \operatorname{int} K_1$, and P_{K_1} is continuous sublinear functional (homogen. w.r.t. multiplication by non-negative numbers & satisfies triangle inequality). Additionally

$$P_{K_1}(x) = 0 \iff x = 0$$

Define $h: K_1 \to \overline{B_1(0)}$: $\quad h(x) = \begin{cases} P_{K_1}(x)\dfrac{x}{\|x\|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

It follows that h is a continuous mapping of the convex set K_1 onto the unit ball B_1, which maps the boundary of K_1 to the boundary of $B_1(0)$; i.e,

$$\|h(x)\| = 1 \quad \text{for } x \in K_1 \setminus \operatorname{int} K_1 \equiv \partial K_1$$

Since $P_{K_1}(x) = 0 \iff x \Rightarrow h(x) = 0 \iff x = 0$

Let $h(x) = h(y)$, $x, y \in K_1$ & $x \neq y$. Then, $\exists \lambda > 0$ s.t. $y = \lambda x$.

Since $P_{K_1}(y) = \lambda P_{K_1}(x)$

$$h(x) = h(y) \Rightarrow P_{K_1}(x)\frac{x}{\|x\|} = P_{K_1}(y)\frac{y}{\|y\|} = \lambda P_{K_1}(x)\frac{x}{\|x\|} \Rightarrow \lambda = 1;$$

therefore, $x = y$. So $\exists$ inverse h^{-1} mapping $\overline{B_1(0)}$ to K_1 (clearly continuous); i.e, h is a homeomorphism.

Let $g = h \circ f_1 \circ h^{-1}$; then, $g : \overline{B_1(0)} \to \overline{B_1(0)}$ is continuous and by Brouwer's FP. $\exists y_0 \in \overline{B_1(0)}$

s.t. $y_0 = g(y_0) = (h \circ f_1 \circ h^{-1})(y_0) \Leftrightarrow h^{-1}(y_0) = f_1(h^{-1}(y_0))$

$\Rightarrow h^{-1}(y_0) \in K_1$ is a fixed point of f_1 on K_1 $\square$

<u>Theorem 1.14</u> Let X be a finite dimensional linear space and f a continous mapping defined on a closed, convex, and bounded subset $K \subset X$ mapping K to itself; i.e., $f(x) \in K$ for all $x \in K$. Then, there exists a fixed point of f in K; i.e, $\exists \bar{x} \in K$ s.t. $\bar{x} = f(\bar{x})$.

<u>Remarks</u>

a) 1D: $f : [a,b] \to [a,b]$

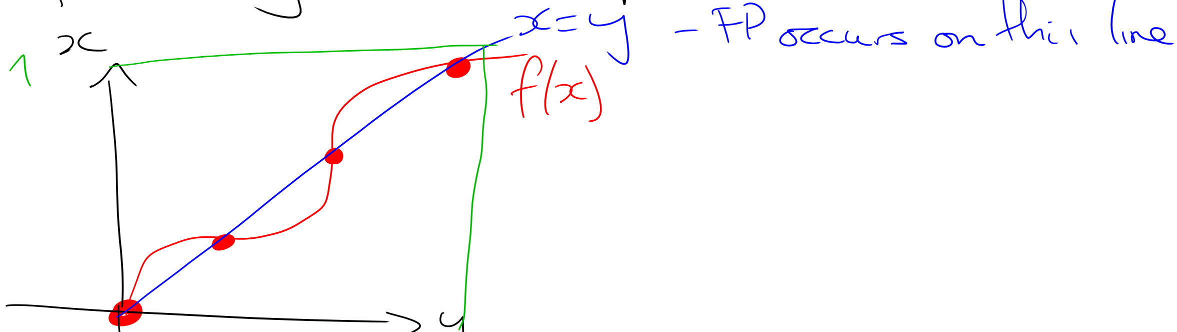
Define $g(x) = f(x) - x$ then

$$g(a) = f(a) - a \geq a - a = 0$$
$$g(b) = f(b) - b \leq b - b$$
$$\Rightarrow g(b) \leq 0 \leq g(a)$$

Intermediate value theorem: $\exists \xi \in [a,b]$ with $g(\xi) = 0 \Leftrightarrow f(\xi) = \xi$

Brouwer's FP theorem $\Leftrightarrow$ intermediate value theorem

b) Fixed point may not be unique



c) FP iteration $x_{n+1} = T(x_n)$, $n \geq 0$ may not converge

d) Does not hold in infinite dimensions - Kakutani's Counter-example

Theorem 1.15 (Schauder-Tychonoff FP Theorem)

Let K be a non-empty, convex, and compact subset of the metric space X. Then, any continuous mapping $T: K \to K$ has at least one fixed point in K.

Theorem 1.16 (Schauder FP Theorem)

Let K be a non-empty, bounded, closed, and convex subset of a Banach space X. Then, any continuous, compact mapping $T: K \to K$ has at least one fixed point in K.

Proof

T is compact $\Rightarrow \forall n \in \mathbb{N}$, $\exists x_1, \ldots, x_N \in K$ s.t.

$$T(K) \subset \bigcup_{i=1}^{N} B_{\frac{1}{n}}(T(x_i))$$

($T(K)$ precompact $\Leftrightarrow$ totally bounded for Banach spaces)

Set $y_i = T(x_i)$ and $x \in K$

$$a_i(x) := \max\left(\tfrac{1}{n} - \|T(x) - y_i\|, 0\right) \text{ for } i = 1, \ldots, N$$

Note
- $a_i \geq 0$ & a_i continuous
- $\sum_{i=1}^{N} a_i(x) > 0$ for any $x \in K$

Define
$$\lambda_i(x) = \frac{a_i(x)}{\sum_{j=1}^{N} a_j(x)}, \quad 1 \leq i \leq N$$

Then, λ_i - continuous

$$\left.\begin{array}{l} 0 \leq \lambda_i \leq 1 \quad \forall x \in K \\ \sum_{i=1}^{N} \lambda_i(x) = 1 \end{array}\right\} \text{Partition of unity.}$$

Define $T_n : K \to K_n := \{ \sum_{i=1}^{N} \alpha_i y_i : 0 \le \alpha_i \le 1, \sum_{i=1}^{N} \alpha_i = 1 \}$

$\subset \operatorname{Span}\{y_1, \dots, y_N\}$

$$T_n(x) = \sum_{i=1}^{N} \lambda_i(x) y_i$$

Then, $T_n(K) = \{ \sum_{i=1}^{N} \lambda_i(x) y_i , x \in K \} \subset K_n$

For $x \in K$

$$\| T_n(x) - T(x) \| = \| \sum_{i=1}^{N} \lambda_i(x)(y_i - T(x)) \|$$

$$\le \sum_{i=1}^{N} \lambda_i(x) \| y_i - T(x) \|$$

$$\le \frac{1}{n} \sum_{i=1}^{N} \lambda_i(x) = \frac{1}{n} \xrightarrow[\text{uniformly}]{n\to\infty} 0$$

Define $\tilde{T}_n := T_n|_K$ $\quad \tilde{T}_n : K_n \to K_n$ (K_n closed, convex hull $\Rightarrow$ bounded)

$\Rightarrow \exists x_n \in K_n$ s.t. $\tilde{T}_n(x_n) = x_n$ (Th 1.14)

As X is complete & K closed $\Rightarrow \{x_n\} \in K$ s.t. $x_n \xrightarrow{n\to\infty} \bar{x} \in K$

$$\| T(\bar{x}) - x_n \| = \| T(\bar{x}) - T_n(x_n) \|$$

$$\le \underbrace{\| T(\bar{x}) - T(x_n) \|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\| T(x_n) - T_n(x_n) \|}_{\le \frac{1}{n} \to 0 \text{ uniformly}}$$

$\Rightarrow \| T(\bar{x}) - x_n \| \to 0$ & $x_n \to \bar{x}$ as $n \to \infty$

$\Rightarrow T(\bar{x}) = \bar{x}$ $\square$

<u>Lemma 1.17</u> X, Y, Z Banach spaces. $T : X \to Y$ is compact and continuous, and $S : Y \to Z$ is continuous. Then, $S \circ T : X \to Z$ is compact and continuous.

<u>Remark</u> If T continuous, and S compact and continuous; Then, $S \circ T$ is compact and continuous.

Proof T, S continuous $\Rightarrow S \circ T$ continuous

$\{x_j\} \subset X$ bounded sequence $\Rightarrow \{T(x_j)\}$ has convergent subseq

$$T(x_{j'}) \xrightarrow{j' \to \infty} y^* \in Y$$

$$\Rightarrow (S \circ T)(x_{j'}) \xrightarrow{j' \to \infty} S(y^*)$$

$$\Rightarrow \{(S \circ T)(x_{j'})\} \xrightarrow{j' \to \infty} S(y^*) \text{ has convergent subsequence}$$

$\Rightarrow S \circ T$ compact.

Theorem 1.18 (Schaefer's F.P. Theorem)

X is a Banach space. $T : X \to X$ continuous & compact.

Moreover, suppose that

$$\bigcup_{0 \le \lambda \le 1} \{x \in X : \lambda T(x) = x\} =: M$$

is bounded. Then T has a fixed point in X.

Proof M bounded $\Rightarrow \exists R > 0$ s.t. $\|x\| < R \;\; \forall x \in M$

Define $\tilde{T} : X \to X$

$$x \longmapsto \begin{cases} \dfrac{T(x)}{\|T(x)\|} \min(R, \|T(x)\|), & T(x) \neq 0 \\ 0 & T(x) = 0 \end{cases}$$

$\tilde{T}$ composition of compact operator T and continuous mapping $\varphi : y \longmapsto \begin{cases} \frac{y}{\|y\|} \min(R, \|y\|) & y \neq 0 \\ 0 & y = 0 \end{cases}$

$\Rightarrow \tilde{T} = \varphi \circ T$ is compact (Lemma 1.17)

Set $K := \underbrace{\operatorname{conv}\big(\underbrace{\overline{\tilde{T}(B_R(0))}}_{\subset X \text{ bounded}}\big)}_{\text{compact}}$ — compact, convex & bounded

$(\tilde{T} : K \to K)$

Schauder FP: $\exists \bar{x} \in K$ s.t $\tilde{T}(\bar{x}) = \bar{x}$

Suppose $\|T(x)\| \geq R$

$$\Rightarrow \bar{x} = \tilde{T}(\bar{x}) = \frac{T(\bar{x})}{\|T(\bar{x})\|} R = \underbrace{\frac{R}{\|T(\bar{x})\|}}_{\in(0,1)} T(\bar{x})$$

$$\Rightarrow \bar{x} \in M \Rightarrow \|\bar{x}\| < R$$

but $\|\bar{x}\| = \frac{R}{\|T(\bar{x})\|} \|T(\bar{x})\| = R$ - contradiction

$$\Rightarrow \|T(x)\| < R \Rightarrow \bar{x} = \tilde{T}(\bar{x}) = T(\bar{x}) \Rightarrow \bar{x} \text{ F.P of } T. \quad \square$$