2 - Monotone & Potential Operators

Consider X as real Hilbert space, or reflexive Banach space with dual X^*.

2.1 - Basic Concepts & Definitions

Definition 2.1 We call $A: X \to X^*$

1) monotone if $\forall u,v \in X$
$$\langle Au - Av, u-v \rangle \geq 0$$
2) strictly monotone if $u,v \in X, u \neq v$
$$\langle Au - Av, u-v \rangle > 0$$
3) strongly monotone if $\exists$ positive constant M s.t
$$\langle Au - Av, u-v \rangle \geq M \|u-v\|^2 \quad \forall u,v \in X$$
4) uniformly monotone if $\exists$ an increasing function $a: \mathbb{R}_+ \to \mathbb{R}_+$, $a(0)=0$ such that $\forall u,v \in X$
$$\langle Au - Av, u-v \rangle \geq a(\|u-v\|)\|u-v\|$$
5) α-monotone (α-monotone) if there exists a strictly increasing function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\forall u,v \in X$
$$\langle Au - Av, u-v \rangle \geq (\alpha(\|u\|) - \alpha(\|v\|))(\|u\| - \|v\|)$$

Definition 2.1 (Coercive) We call $A: X \to X^*$

1) coercive if $\displaystyle\lim_{\|u\| \to +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty$

Equivalently: $\exists$ a function $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\displaystyle\lim_{r \to +\infty} \gamma(r) = +\infty$ and $\langle Au, v \rangle \geq \gamma(\|u\|)\|u\| \quad \forall u \in X$

2) weakly coercive if
$$\lim_{\|u\| \to +\infty} \|Au\| = +\infty$$

Definition 2.3 We say that $A: X \to X^*$ meets the conditions

(S) if $\{u_n \rightharpoonup u, \langle Au_n - Au, u_n - u\rangle \to 0\} \Rightarrow u_n \to u$

(M) if $\{u_n \rightharpoonup u, Au_n \rightharpoonup b, \limsup_{n\to\infty} \langle Au_n, u_n\rangle \leq \langle b, u\rangle\}$
$$\Rightarrow Au = b, \ b \in X^*$$

(M_0) if $\{u_n \rightharpoonup u, Au_n \rightharpoonup b, \limsup_{n\to\infty} \langle Au_n, u_n\rangle = \langle b, u\rangle\}$
$$\Rightarrow Au = b, \ b \in X^*$$

Remark Monotone operator can be extended to $T: X \to Y$ where Y is linearly isometric isomorphic to X^*. Denoting the isometric isomorphism as $I_Y \in \mathcal{L}(X^*, Y)$ and define $A: X \to X^*$ by $A = I_Y^{-1} T: X \to X^*$. We say that $T: X \to X$ is monotone if A is monotone. This extension can be found when Y represents continuous linear functions from X^*. Analogously, all above can be defined.

Lemma 2.4 The following hold for $A: X \to X^*$

a) strongly monotone $\Rightarrow$ α-monotone $\quad (\alpha(s) = Ms)$

b) strongly monotone $\Rightarrow$ uniformly monotone

c) uniformly monotone $\Rightarrow$ strictly monotone

d) uniformly monotone $\Rightarrow$ (S) holds

e) uniformly monotone $\Rightarrow$ coercive

f) strictly monotone $\Rightarrow$ monotone

g) α-monotone & $\lim_{s\to\infty} \alpha(s) = +\infty$ $\Rightarrow$ coercive

h) α-monotone $\Rightarrow$ monotone

i) X strictly convex : α-monotone $\Rightarrow$ strictly monotone

j) X uniformly convex: α-monotone $\Rightarrow$ strictly monotone & (S) holds

<u>Remark</u> Set $\mathcal{M}$ of all monotone operators (or strongly, strictly, uniformly, or α- monotone) form a cone; i.e., for $\forall A_1, A_2 \in \mathcal{M}$ then $A_1 + A_2$ and $t>0$, $tA_1 \in \mathcal{M}$.

- Additionally, sum of two operators from different "monotone" sets has "stronger monotonicity"; e.g. sum of monotone & strictly monotone operators is strictly monotone.
- Sum of coercive operator and monotone (strictly monotone) operator is coercive.
- If A uniformly monotone & (S) holds then

$$\langle Au_n - Au, u_n - u\rangle \to 0 \Rightarrow u_n \to u$$

<u>Definition 2.5</u> We call $A: X \to X^*$

1) bounded if there exists an $M_1 : \mathbb{R}_+ \to \mathbb{R}_+$ s.t.

$$\|Au\| \leq M_1(\|u\|)$$

2) locally bounded if for each $u \in X$ $\exists \varepsilon = \varepsilon(u) > 0$ and $M = M(u)$ s.t.

$$\|Av\| \leq M \quad \forall v \text{ such that } \|u - v\| \leq \varepsilon$$

3) continuous if

$$u_n \to u \Rightarrow Au_n \to Au \quad u, u_n \in X$$

4) strongly continuous if

$$u_n \rightharpoonup u \Rightarrow Au_n \to Au \quad u, u_n \in X$$

(totally continuous).

5) weakly continuous if

$$u_n \rightharpoonup u \Rightarrow Au_n \rightharpoonup Au \qquad u, u_n \in X$$

6) completely continuous if for $M \subset X$

A continuous & (M bounded $\Rightarrow$ $A(M)$ compact)

(sometimes, $A(M)$ precompact / relatively compact).

7) hemicontinuous (weakly cont. on lines) if

$$t_n \in \mathbb{R}, t_n \to 0 \Rightarrow A(u+t_n v) \rightharpoonup Au \quad \forall u, v \in X$$

Equivalently: function $\varphi_{u,v}(t) := \langle A(u+tv), w \rangle$ continuous on $[0,1]$ $\forall u, v, w \in X$

8) radially continuous if function

$$\varphi_{u,v}(t) := \langle A(u+tv), v \rangle \text{ continuous on } [0,1] \ \forall u, v \in X$$

9) demicontinuous if

$$u_n \to u \Rightarrow Au_n \rightharpoonup Au \qquad u, u_n \in X$$

10) uniformly continuous if $\exists$ a continuous function

$M : \mathbb{R}_+ \to \mathbb{R}_+$, $M(0) = 0$ such that

$$\|Au - Av\| \leq M(\|u-v\|) \qquad \forall u, v \in X$$

11) Lipschitz continuous if $\exists L > 0$ such that

$$\|Au - Av\| \leq L\|u-v\| \qquad \forall u, v \in X$$

12) continuous on lines if

$$t_n \in \mathbb{R}, t_n \to 0 \Rightarrow A(u+t_n v) \to Au \quad \forall u, v \in X$$

13) continuous on finite dimensional subspaces if

$X_n \subset X$, $\dim X_n < +\infty \Rightarrow A|_{X_n} : X_n \to X_n^*$ continuous

where $A|_{X_n}$ is A restricted to X_n

Equivalently: $X_n \subset X$, $\dim X_n < +\infty$, $\{u_j\} \subset X_n$,
$$u_j \to u \Rightarrow Au_j \to Au$$

14) **bounded Lipschitz continuous** if there exists an increasing function $M : \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|Au - Av\| \le M(R)\|u - v\|, \quad R = \max(\|u\|, \|v\|) \quad \forall u, v \in X_0$$

Lemma 2.6 $A : X \to X^*$

$$\begin{tikzcd}
\boxed{\text{bounded Lipschitz cont.}} \arrow[d, Rightarrow] & \boxed{\text{strongly cont.}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{weakly cont.}} \arrow[d, Rightarrow] \\
\boxed{\text{bounded}} & \boxed{\text{completely cont.}} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{demicontinuous}} \arrow[d, Rightarrow] \\
\boxed{\text{uniformly cont.}} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \boxed{\text{continuous}} \arrow[ur, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{cont. of finite dim. subspaces}} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
\boxed{\text{Lipschitz cont.}} \arrow[u, Rightarrow] & \boxed{\text{cont. on lines}} \arrow[r, Rightarrow] & \boxed{\text{hemicontinuous}} \arrow[d, Rightarrow] \\
\boxed{\text{locally bounded}} & \boxed{\text{demicontinuous}} \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[ur, Rightarrow] & \boxed{\text{radially cont.}}
\end{tikzcd}$$

Lemma 2.7 Let $A : X \to X^*$ be monotone. Then,

1) A is locally bounded
2) Let $K \subset X$ and, for all $u \in K$,
$$\|u\| \le M_1 \quad \text{and} \quad \langle Au, u\rangle \le M_2$$
where M_1, M_2 are constants. Then, there exists a constant M such that
$$\|Au\| \le M \quad \forall u \in K$$

3) Function $\varphi_{u,v}(t) := \langle A(u+tv), v\rangle$ is non-decreasing on $[0,1]$ $\forall u,v \in X$. Conversely if $\varphi_{u,v}(t) := \langle B(u+tv), v\rangle$ is non-decreasing on $[0,1]$ $\forall u,v \in X$ then $B: X \to X^*$ monotone.

4) The following are equivalent

a) A is radially continuous

b) $\langle b - Av, u-v\rangle \geq 0$ $\forall v \in X \Rightarrow Au = b$, $b \in X^*$

c) A satisfies (M)

d) A is demicontinuous

e) K is dense in X, $\langle b - Av, u-v\rangle \geq 0$ $\forall v \in K \Rightarrow Au = b$, $b \in X^*$

f) A is hemicontinuous

5) (Minty) If A is hemicontinuous (or rad. cont. or demicont.) and $u \in X$, then

$$\langle Au, v-u\rangle \geq 0 \; \forall v \in X \Leftrightarrow \langle Av, v-u\rangle \geq 0 \; \forall v \in X$$

6) Let there exist a Gateaux derivative of A; i.e, there exists a linear operator $A': X \to \mathcal{L}(X, X^*)$ such that for all $u, v, h \in X$

$$\lim_{t \to 0} \langle A(u+th) - Au, v\rangle = \langle A'(u)h, v\rangle.$$

Assume functional $\langle A'(u+tv)v, v\rangle$ is continuous on t in the interval $[0,1]$ for every $u, v \in X$; then, $\langle A'(u)v, v\rangle \geq 0$. Conversely, if there exists a Gateaux derivative of $B: X \to X^*$ and the function $\langle B'(u+tv)v, v\rangle$ is continuous on the interval $[0,1]$ $\forall u,v \in X$ and $\langle B'(u)v, v\rangle \geq 0$; then, B is monotone.

7) If A is linear then it is continuous.

8) Let K be a bounded subset of X for which it holds that $\langle Au, u\rangle \leq M_1$ for all $u \in K$, where $M_1 > 0$ is a constant. Then, there exists a constant $M > 0$ s.t. $\|Au\| \leq M$ $\forall u \in K$.

2.2 Pseudomonotone Operator

Pseudomonotone operators combine continuity & monotonicity.

Definition 2.8 We call the operator $A: X \to X^*$ pseudomonotone if for all $v \in X$ it satisfies

$$\{u_n \rightharpoonup u, \limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0\}$$

$$\Rightarrow \liminf_{n\to\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle$$

Definition 2.9 We say that $A: X \to X^*$ satisfies

$(S)_+$ if the following holds

$$\{u_n \rightharpoonup u, \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \leq 0\} \Rightarrow u_n \to u$$

$(S)_0$ if the following holds

$$\{u_n \rightharpoonup u, Au_n \rightharpoonup b, \langle Au_n, u_n\rangle \to \langle b, u\rangle\} \Rightarrow u_n \to u$$

Lemma 2.10 Let operator $A: X \to X^*$ be defined on the reflexive Banach space X. Then, the following holds

1) monotone & hemicontinuous $\Rightarrow$ pseudomonotone
2) demicontinuous & $(S)_+$ holds $\Rightarrow$ pseudomonotone
3) pseudomonotone $\Rightarrow$ (M) holds
4) demicontinuous & $(S)_0$ holds $\Rightarrow$ $(M)_0$ holds
5) strongly continuous $\Rightarrow$ pseudomonotone
6) pseudomonotone & bounded $\Rightarrow$ demicontinuous

7) $(S)_+$ holds $\Rightarrow$ (S) holds
8) (S) holds $\Rightarrow$ $(S)_0$ holds
9) (M) holds $\Rightarrow$ $(M)_0$ holds

<u>Remark</u> If we consider Hilbert spaces with inner product $(\circ,\circ)$ from Reisz-Frechet Theorem on gene form of continuous linear functionals all above terms can be applied to this special case of reflexive Banach spaces with inner product instead of dual relationship, e.g. $A: X \to X$ is strongly monotone if $\exists C>0$ s.t for any $x, y \in X$

$$(Ax - Ay, x-y) \geq C\|x-y\|^2.$$