3 - Differential Equations

3.1 Definitions

- Let $n \in \mathbb{N}$. We define the vector $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^n$ as the multi-index with length $|\alpha| = \sum_{j=1}^{n} \alpha_j$

  It is possible to show that there exists
  $$K = \frac{(n+k)!}{n!\,k!}$$
  different multi-indices of length $|\alpha| \leq k$.

- We define the operator
  $$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$
  and vector function
  $$\delta_k u = (D^\alpha u)_{|\alpha| \leq k} = \left(u, \frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_n}, \dots \frac{\partial^k}{\partial x_n^k}\right)$$

  Remark
  - Don't care about order of differentiation for mixed derivatives; e.g. $\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_2 \partial x_1}$ so exclude repeats (e.g. $\frac{\partial^2 u}{\partial x_2 \partial x_1}$) in $\delta_k u$
  - Order of vector: u, then first derivatives, then second, ... finally kth order. For same order derivatives sort by highest order of x_1 then x_2 etc.

- We also define $\hat{\delta}_k u = (D^\alpha u)_{|\alpha| = k}$ noting that $\delta_k u = (\delta_{k-1} u, \hat{\delta}_k u)$ for $k \geq 1$.
- We denote by $\Omega \subset \mathbb{R}^n$ a bounded measurable domain with Lipschitz continuous boundary

- Denote by $L^p(\Omega)$ the standard Lebesgue space with norm

$$\|v\|_p = \left(\int_\Omega v^p \, dx\right)^{1/p} \qquad 1 \leq p < \infty$$

$$\|v\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|$$

and define the Sobolev space

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\beta u \in L^p(\Omega) \ \forall |\beta| \leq k\}$$

with seminorm and norm

$$|u|_{k,p} = \left(\sum_{|\beta|=k} \|D^\beta u\|_p^p\right)^{1/p} \ \& \ \|u\|_{k,p} = \left(\sum_{j=0}^{k} |u|_{k,p}^p\right)^{1/p}$$

respectively. For $p=2$ we let $H^k(\Omega) = W^{k,2}(\Omega)$.

The set of infinetly smooth functions with compact support in Ω $(C_0^\infty(\Omega))$ is subspace of $W^{k,p}(\Omega)$.

Denote by $W_0^{k,p}(\Omega)$ the completion of this space in $\|\cdot\|_{k,p}$. This space has norm $\|u\|_{k,p,0} = \left(\sum_{|\beta|=k} \|D^\alpha u\|_p^p\right)^{1/p}$

$W_0^{k,p}(\Omega) = \{u \in W^{k,p}(\Omega) : D^\beta u = 0$ on Ω in the trace $|\beta| \leq k-1\}$

- We denote by DF and dF the Fréchet and Gâteaux derivatives of F, respectively.

3.2 General form of nonlinear PDEs

We can write a nonlinear differential equation of order k as $F(x, \delta_k u(x)) = 0$ for $x \in \Omega$

where $k \in \mathbb{N}, k \leq n$ and $F(x, \delta_k u(x))$ is a function defined over $x \in \Omega$.

Definition 3.1 A differential equation of order k is said to be

- linear if it is linear in the unknown function u and all its derivatives of order $\leq k$ with coefficients depending only on independent variables ($x \in \Omega$)
- semilinear if it is linear in the derivatives of order k with coefficients dependent on independent variables only
- quasilinear if it is linear in derivatives of order k with coefficients dependent on independent variables and derivatives of order less than k.
- (fully) nonlinear if it is nonlinear in derivatives of order k, with coefficients dependent on independent variables and derivatives of order k.

3.3 Divergence Form

We can write linear, semilinear, or quasilinear differential equations of order $2k$ in divergence form

$$(1) \quad \sum_{|\alpha| \leq k} D^{\alpha} a_{\alpha}(x, \delta_k u(x)) = f(x) \quad \text{for } x \in \Omega$$

where $k \in \mathbb{N}$, α is an n-dimensional multi-index, $a_{\alpha} = a_{\alpha}(x, \xi)$, $|\alpha| \leq k$, is a function of $n+K$ variables over $x \in \Omega$, $\xi \in \mathbb{R}^K$, and f a function defined over Ω.

Suppose all coefficients a_{α} in (1) have continuous derivatives of order $|\alpha|$; i.e.,

$$a_{\alpha} \in C^{|\alpha|}(\Omega \times \mathbb{R}^k)$$

and $f \in C^0(\Omega)$. Then, $u = u(x)$ over Ω is the classical solution of ① if $u \in C^{2k}(\Omega)$ and u satisfies ①.

3.4 Nemyckii Operators

Definition 3.2 (Caratheodory conditions)

Let $\Omega \subset \mathbb{R}^n$ be non-empty and measurable, and $m \in \mathbb{N}$; then, for the function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ the Caratheodory conditions are

1) $f(\cdot, u)$ is measurable on Ω for all fixed $u \in \mathbb{R}^m$
2) $f(x, \cdot)$ is continuous almost everywhere for $x \in \Omega$

For function $h(x, \xi)$, $x \in \Omega$, $\xi \in \mathbb{R}$ satisfying the Caratheodory conditions we denote that

$$h \in CAR$$

Definition 3.3 (Nemyckii Operators)

Let $\Omega \subset \mathbb{R}^n$ be non-empty and measurable, and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \in \mathbb{N}$, satisfying the Caratheodory conditions; then, for a function $u : \Omega \to \mathbb{R}^m$, $u(x) = (u_1(x), \dots, u_m(x))$ we define a Nemyckii operator $\mathcal{N}$ as

$$\mathcal{N}u(x) = f(x, u(x))$$

Theorem 3.4 (Properties of Nemyckii Operators)

Let $\mathcal{N}$ be a Nemyckii operator, with function f satisfying the Caratheodory conditions; then,

1) $\mathcal{N}$ maps measurable functions to measurable functions

2) Let $1<p,q<\infty$. If the growth condition

$$|f(x,u)| \leq g_1(x) + c(x)\sum_{i=1}^{m} |u_i|^{p/q}$$

where $g_1(x) \in L^q(\Omega)$ and c is a non-negative L^∞-function. is met; then $\mathcal{N}$ is a well-defined, bounded, continuous operator from $[L^p(\Omega)]^m$ to $L^q(\Omega)$.

3) Select $p=2$ and $m=1$. Assume $f(x,u)$ and partial derivative with respect to u $(f_u'(x,u))$ satisfies the Caratheodory conditions, and let $\exists C>0$ such that

$$|f_u'(x,u)| \leq C \quad \text{for } x\in\Omega \text{ and } u\in\mathbb{R};$$

then, the Nemyckii operator $\mathcal{N}: L^2(\Omega)\to L^2(\Omega)$ is Gâteaux-differentiable and

$$d\mathcal{N}(u,v) = f_u'(x,u(x))\,v(x) \quad \forall u,v\in L^2(\Omega)$$

Additionally, the linear mapping $d\mathcal{N}(u,\cdot)$ is bounded and

$$|d\mathcal{N}(u,v)| \leq C\|v\|$$

holds.

4) Let $p>1$ and $m=1$. Assume $f(x,u)$ satisfies Caratheodory conditions, and that partial derivative w.r.t u $(f_u'(x,u))$ satisfies

$$|f_u'(x,u)| \leq g(x) + b|u|^{p-2} \quad \text{for } x\in\Omega \text{ and } u\in\mathbb{R}$$

where $g\in L^{\frac{p}{p-2}}(\Omega)$ and $b>0$; then, the Nemyckii operator $\mathcal{N}: L^p(\Omega)\to L^q(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$ is Fréchet-diffferentiable and

$$D\mathcal{N}(u,v) = f_u'(\cdot,u(\cdot))\,v(\cdot) \quad \forall u,v\in L^p(\Omega)$$

Proof We outline the proof for $n=1$ only.

1) From theory of measurable functions we know that $f(x,u): \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable if $f(x,u)$ is continuous in u for nearly all $x \in \Omega$ and measurable as a function of $x \in \Omega$ for all $u \in \mathbb{R}$.

2) Let $1 < p, q < \infty$, the growth condition

$$|f(x,u)| \leq g_1(x) + c(x)|u|^{p/q}$$

where $g_1(x) \in L^q(\Omega)$ and c is a non-negative L^∞-function; then, using Minkowski/triangle inequality

$$\|\mathcal{N}u\|_q \leq \left(\int_\Omega (g_1 + c|u|^{p/q})^q dx \right)^{1/q}$$

$$\leq \|g_1\|_q + \|c\|_\infty \left(\int_\Omega (|u|^{p/q})^q dx \right)^{1/q}$$

$$= \|g_1\|_q + \|c\|_\infty \|u\|_p^{p/q} < \infty$$

This implies that $\mathcal{N}$ bounded operator from $L^p(\Omega)$ to $L^q(\Omega)$.

From continuity, suppose sequence $\{u_n\}$ converges in $L^p(\Omega)$ to $u \in L^p(\Omega)$; then, there exists a subsequence $\{u_k\}$ which converges to u almost everywhere. Since $f(x,u)$ is continuous in u then the sequence $\mathcal{N}u_k(x) = f(x,u_k)$ converges to $\mathcal{N}u(x) = f(x,u)$ almost everywhere. We then use Vitali's convergence theorem. Since $u_k(x) \to u(x)$ almost everywhere there exists for every $\varepsilon > 0$ a subdomain $\Omega_\varepsilon \subset \Omega$ with measure $\mu(\Omega_\varepsilon) < \infty$ and real number $\delta > 0$ s.t. for all $A \subset \Omega$ with $\mu(A) < \delta$ it holds

$$\int_A |u_k(x)|^p dx \leq \varepsilon \quad \& \quad \int_{\Omega \setminus \Omega_\varepsilon} |u_k(x)|^p dx < \varepsilon$$

uniformly for $k \in \mathbb{N}$. By Minkowski & growth condition

$$\left(\int_A |\mathcal{N} u_k(x)|^q dx\right)^{1/q} \le \left(\int_A |g_1(x)|^q dx\right)^{1/q} + \|C\|_\infty \, \varepsilon^{1/q}$$

Similar holds for integrals over Ω_ε and $\mathcal{N} u_k$ meets requirements of Vitali's convergence theorem $\Rightarrow$ $\mathcal{N} u_k$ converges to $\mathcal{N} u$ in $L^q(\Omega)$.

3) From mean value theorem $\exists s \in (0,1)$ such that

$$\frac{1}{t}\left[\mathcal{N}(u+tv) - \mathcal{N}(u)\right] - d\mathcal{N}(u,v) = \left(f_u'(\cdot, u+stv) - f_u'(\cdot, v)\right)$$

Let $t \to 0$, then $stv \to 0$ and

$$f_u'(x, u+stv) \to f_u'(x,u) \quad \text{a.e. for } x \in \Omega$$

Therefore, for all $u, v \in L^2(\Omega)$

$$\lim_{t \to 0} \frac{1}{t}\left|\mathcal{N}(u+tv) - \mathcal{N}(u)\right| = d\mathcal{N}(u,v) = f_u'(x, u(x))\, v(x)$$

Bound follows trivially.

4) By integration of condition on $f_u'(x,u)$ we obtain

$$|f(x,u)| \le g(x) + b_1 |u|^{p-1}$$

$$\left[\frac{1}{(p/q)} + \frac{1}{\left(\frac{p}{p-q}\right)} = 1 \qquad \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p-1 = p/q\right]$$

$$\Rightarrow \|f(x,u)\|_q \le \left(\int_\Omega \left(g(x)|u| + b|u|^{p/q}\right)^q\right)^{1/q}$$

$$\le \left(\int_\Omega \left(|g(x)|^q\right)^{\frac{p}{p-q}}\right)^{\frac{(p-q)}{qp}} \left(\int_\Omega \left(|u(x)|^q\right)^{\frac{p}{q}}\right)^{1/p}$$

$$+ \|b_1\|_\infty \left(\int_\Omega \left(|u|^{p/q}\right)^q\right)^{1/q} \quad \text{(Hölder)}$$

$$\le \|g\|_{\frac{p}{p+2}} \|u\|_p + \|b_1\|_\infty \|u\|_p^{p/q} < \infty$$

$$\left(\text{as } q\frac{p}{p-q} = \frac{p}{\frac{p}{q} - 1} = \frac{p}{p-2}\right)$$

$\Rightarrow f \in L^q(\Omega)$ for all $u \in L^p(\Omega)$

For any $u, v \in L^p(\Omega)$ define

$$H(u,v) = \|\mathcal{N}(u+v) - \mathcal{N}(u) - f_u'(\cdot, u) v\|_q$$

where $\mathcal{N}(u+v)(x) = f(x, u(x) + v(x))$.

By mean value theorem $\exists s \in (0,1)$ such that

$$f(x,u+v) - f(x,u) = f_u'(x, u+sv)\, v$$

and, therefore, $H(u,v) = \left(\int_\Omega |v(x)\, w(x)|^q dx\right)^{1/q}$

where $w(x) = f_u'(x, u(x)+sv) - f_u'(x, u(x))$.

From Hölder's inequality

$$H(u,v) \le \|v\|_p \|w\|_{\frac{p}{p-2}}.$$

From point 1 Nemyckii operator generated by $f_u'(\cdot,\cdot)$ is continuous from $L^p(\Omega)$ to $L^{\frac{p}{p-2}}(\Omega)$, and, therefore

$$\|w\|_{\frac{p}{p-2}} = \|f_u'(\cdot, u+sv) - f_u'(\cdot, u)\|_{\frac{p}{p-2}} \to 0 \text{ as } \|v\|_p \to 0$$

Therefore, $\dfrac{H(u,v)}{\|v\|_p} \le \|w\|_q \to 0$ as $\|v\|_p \to 0$;

hence, there exists $A = f_u'(\cdot, u)$ such that

$$\lim_{\|v\|_p \to 0} \frac{\|\mathcal{N}(u+v) - \mathcal{N}(u) - Av\|_q}{\|v\|_p} = 0$$

$\Rightarrow \mathcal{N}$ is Fréchet-differentiable with Fréchet derivative Av

$\square$