Theorem 3.19 Let the coefficients $a_\alpha(x,\xi)$ meet the conditions

1) $a_\alpha \in CAR^\alpha(p)$
2) $\dfrac{\partial a_\alpha(x,\xi)}{\partial \xi_\beta} \equiv a_{\alpha\beta}(x,\xi) = a_{\beta\alpha}(x,\xi) = \dfrac{\partial a_\beta(x,\xi)}{\partial \xi_\alpha}$
   for all multi-indices α,β $|\alpha| \le k$, $|\beta| \le k$
3) $\xi_\alpha a_{\alpha\beta}(x,\xi) \in CAR^\alpha(p)$ for all multi-indices α,β, $|\alpha| \le k$, $|\beta| \le k$
4) $\sum_{|\alpha| \le k} (a_\alpha(x,\xi) - a_\alpha(x,\eta))(\xi_\alpha - \eta_\alpha) \ge 0 \quad \forall \xi,\eta \in \mathbb{R}^K$
5) $\sum_{|\alpha| \le k} \xi_\alpha a_\alpha(x,\xi) \ge c_1 \sum_{|\alpha| = k} |\xi_\alpha|^p + c_2 |\xi_{(0,\dots,0)}|^p - c_3$
   where $c_1 > 0$, $c_2 > 0$ and $c_3 \ge 0$ are constants.

Then, there exists at least one weak solution of the boundary value problem $(\mathcal{A}, V, Q)$. Furthermore, if equality in condition 4 holds only when $\xi = \eta$; then, the solution is unique.

Proof Theorems 3.13, 3.15 and 3.17 prove T is monotone and coercive, and Corollary 3.16 is strictly monotone under the stronger conditions. So if we can show T is a potential operator we can apply Theorem 2.27 to show that $Tu = 0$; i.e, for all $u, v \in V$

5 $\displaystyle \lim_{t \to 0} \frac{F(u+tv) - F(u)}{t} = \frac{d}{dt} F(u+tv)\Big|_{t=0} = \langle Tu, v \rangle$

$$= \langle A(u+\varphi), v \rangle - (f, v)_Q - \langle g, v \rangle_V$$

$$= \sum_{|\alpha|\le k} \int_\Omega a_\alpha(x, \delta_k u(x) + \delta_k \varphi(x)) D^\alpha v(x) dx - \langle f, v\rangle_Q - \langle g, v\rangle_V.$$

On space V look for functional F in form

$$F(u) = \int_\Omega b(x, \delta_k u(x) + \delta_k \varphi(x)) dx - \langle f, u\rangle_Q - \langle g, v\rangle_V$$

where $b(x,\xi)$ is defined for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^k$. The integral on the right hand side will be finite if $b \in CAR(p+1)$.

By Theorem 3.11 the Nemyckii operator corresponding to the function b is bounded from the space $W^{k,p}(\Omega)$ to the space $L^s(\Omega)$, where $s = 1 + \frac{1}{p}$. As $L^s(\Omega) \subset L^1(\Omega)$ then the integral is finite. Furthermore if $b \in CAR^*(p+1)$ the integral can be shown to be finite using Theorem 3.12

Assume for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^K$ that the following partial derivatives exist

$$\frac{\partial b(x,\xi)}{\partial \xi_\alpha}, |\alpha| \le k \quad \& \quad \frac{\partial b}{\partial \xi_\alpha} \in CAR^*(p) \quad \forall \alpha, |\alpha| \le k.$$

Then, the order of integration and differentiation can be exchanged (Lebesgue integral theory) and, hence,

$$a(u,v) \equiv \frac{d}{dt} F(u+tv)\Big|_{t=0}$$

$$= \int_\Omega \frac{\partial}{\partial t} b(x, \delta_k u(x) + \delta_k \varphi(x) + t\delta_k v(x))\Big|_{t=0} dx - \langle f, v\rangle_Q - \langle g, v\rangle_V$$

6 $$= \int_\Omega \sum_{|\alpha|\le k} \frac{\partial}{\partial \xi_k} b(x, \delta_k u(x) + \delta_k \varphi(x)) D^\alpha v(x) dx - \langle f, v\rangle_Q - \langle g, v\rangle_V$$

By definition of Gâteaux derivative if, for every fixed $u \in V$, $G(u,v)$ is linear and continuous functional in $v \in V$ then $G(u,v) = \langle F'(u), \cdot \rangle$ is a Gâteaux derivative of the functional F at point u.
Linearity is immediately obvious from above formula, we can show continuity if

$$\frac{\partial b}{\partial \xi_\alpha} \in CAR(p) \quad \forall \alpha, |\alpha| \leq k.$$

By Theorem 3.11

$$\int_\Omega \left| \frac{\partial b}{\partial \xi_\alpha}(x, \delta_k u(x) + \delta_k \varphi(x)) D^\alpha v(x) \right| dx$$
$$\leq C_\alpha \left(1 + \sum_{|\nu| \leq k} \| D^\nu u + D^\nu \varphi \|_p^{p-1} \right) \| D^\alpha v \|_p$$
$$\leq C_\alpha \left(1 + \sum_{|\nu| \leq k} \| D^\nu u + D^\nu \varphi \|_p^{p-1} \right) \| v \|_{k,p}$$

Hence, there exists a constant $C > 0$ such that for all $u, v \in V$

$$|G(u,v)| \leq \sum_{|\alpha| \leq k} \int_\Omega \left| \frac{\partial b}{\partial \xi_\alpha}(x, \delta_k u(x) + \delta_k \varphi(x)) D^\alpha v(x) \right| dx + (\|f\|_Q + \|g\|_V) \|v\|_{k,p}$$

$$\leq C \left(1 + \sum_{|\nu| \leq k} \| D^\nu u + D^\nu \varphi \|_p^{p-1} \right) \| v \|_{k,p}$$

and thus continuous

$$\sup_{\|v\|_{k,p} = 1} |G(u,v)| < \infty \quad \forall u \in V.$$

Similarly, by Theorem 3.12 it is possible to show continuity if $\frac{\partial b}{\partial \xi_\alpha} \in CAR^*(p)$ $\forall \alpha, |\alpha| \leq k$.

If we compare (5) and (6) we require that for the function $b(x,\xi)$ $\frac{\partial b(x,\xi)}{\partial \xi_\alpha} = a_\alpha(x,\xi)$ for all $|\alpha|\le k$.

Then, T is a potential operator with potential F.

Define $b(x,\xi)$ such that

$$b(x,\xi) = \int_0^1 \sum_{|\alpha|\le k} \xi_\alpha a_\alpha(x, t\xi)\, dt.$$

From assumptions 2 & 3 and continuous dependence of integral on its parameter then we obtain that $b \in CAR$ and additionally $b \in CAR^{\sim}(p+1)$.

We now only need to show that for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^K$ that

$$\frac{\partial b(x,\xi)}{\partial \xi_\beta} = a_\beta(x,\xi)$$

$$\frac{\partial b(x,\xi)}{\partial \xi_\beta} = \int_0^1 a_\beta(x,t\xi)dt + \int_0^1 t \sum_{|\alpha|\le k} a_{\alpha\beta}(x,t\xi)dt$$

$$= \int_0^1 a_\beta(x,t\xi)dt + \int_0^1 t \frac{d}{dt}(a_\beta(x,t\xi))dt$$

$$= \int_0^1 a_\beta(x,t\xi)dt + a_\beta(x,\xi) - \int_0^1 a_\beta(x,t\xi)dt$$

$$= a_\beta(x,\xi)$$

□

<u>Theorem 3.20</u> Let the coefficients $a_\alpha(x,\xi)$ meet the conditions

1) $a_\alpha \in CAR^{\sim}(p)$

2) $\sum_{|\alpha|\le k} (a_\alpha(x,\eta,\xi) - a_\alpha(x,\eta,\hat{\xi}))(\xi_\alpha - \hat{\xi}_\alpha) \ge 0$

for all $\eta \in \mathbb{R}^{\tilde{K}}$, $\xi, \hat{\xi} \in \mathbb{R}^{K-\tilde{K}}$, where $\tilde{K} = \frac{(N+k-1)!}{N!(k-1)!}$ is the number of multi-indices of length $|\alpha| \le k-1$.

3) $\sum_{|\alpha|\le k} \xi_\alpha a_\alpha(x,\xi) \geq C_1 \sum_{|\alpha|=k} |\xi_\alpha|^p + C_2 |\xi_{(0,\dots,0)}|^p - C_3$

where $C_1, C_2 > 0$ and $C_3 \geq 0$ are constants.

Then, there exists at least one weak solution of the boundary value problem $(\mathcal{A}, V, Q)$.

<u>Proof</u> We prove this theorem by proving $Tu=0$ has a solution by Leray-Lions (Theorem 3.18). Theorems 3.13, 3.14, and 3.17 prove V is reflexive and T is bounded, demicontinuous, and coercive. It remains to show there exists a bounded map Φ from $V \times V$ to V^* such that

(a) for each $u \in V$ $\quad \Phi(u,u) = Tu$

(b) $\forall u, w, h \in V$ and any sequence $\{t_n\}_{n=1}^{\infty} \in \mathbb{R}$ such that $t_n \to 0$

$$\Phi(u + t_n h, w) \rightharpoonup \Phi(u, w)$$

(c) $\forall u, w \in V$ monotonicity in the first term is met; i.e.,

$$\langle \Phi(u,u) - \Phi(w,u), u - w \rangle \geq 0$$

(d) for $u_n \rightharpoonup u$ and $\lim_{n\to\infty} \langle \Phi(u_n, u_n) - \Phi(u, u_n), u_n - u \rangle = 0$; then, for any $w \in V$

$$\Phi(w, u_n) \rightharpoonup \Phi(w, u)$$

(e) if $w \in V$, $u_n \rightharpoonup u$, $\Phi(w, u_n) \rightharpoonup z$; then

$$\lim_{n\to\infty} \langle \Phi(w, u_n), u_n \rangle = \langle z, u \rangle.$$

Define $\Phi: V\times V \to V^*$ by

$$\langle \Phi(u,w),v\rangle = \int_\Omega \sum_{|\alpha|=k} a_\alpha(x,\delta_{k-1}w-\delta_{k-1}\varphi,\hat{\delta}_k u+\hat{\delta}_k\varphi)D^\alpha v\,dx$$
$$+\int_\Omega \sum_{|\alpha|\le k-1} a_\alpha(x,\delta_k w+\delta_k\varphi)D^\alpha v - \langle f,v\rangle_Q - \langle g,v\rangle_V$$

From definition of Tu ($\langle Tu,v\rangle$) clearly

$$\Phi(u,u)=Tu \qquad \forall u\in V$$

which proves condition (a).

By continuity of Nemydkii operator (Theorem 3.11) it follows that

$$a_\alpha(x,\delta_{k-1}w-\delta_{k-1}\varphi, u+t_n\delta_k h+\delta_h\varphi)$$
$$\to a_\alpha(x,\delta_{k-1}w-\delta_{k-1}\varphi, u+\delta_h\varphi) \text{ in } L^r(\Omega)$$

as $t_n\to 0$ and thus it holds that

$$\lim_{t_n\to 0}\langle\Phi(u+t_n h,w),v\rangle$$
$$=\lim_{t_n\to 0}\int_\Omega\sum_{|\alpha|=k} a_\alpha(x,\delta_{k-1}w-\delta_{k-1}\varphi,\hat{\delta}_k u+t_n\hat{\delta}_k h+\hat{\delta}_k\varphi)D^\alpha v\,dx$$
$$+\int_\Omega\sum_{|\alpha|\le k-1} a_\alpha(x,\delta_k w+\delta_k\varphi)D^\alpha v\,dx-\langle f,v\rangle_Q-\langle g,v\rangle_V$$
$$=\int_\Omega\sum_{|\alpha|\le k} a_\alpha(x,\delta_{k-1}w-\delta_{k-1}\varphi,\hat{\delta}_k u+\hat{\delta}_k\varphi)D^\alpha v\,dx$$
$$+\int_\Omega\sum_{|\alpha|\le k-1} a_\alpha(x,\delta_k w+\delta_k\varphi)D^\alpha v\,dx-\langle f,v\rangle_Q-\langle g,v\rangle_V$$
$$=\langle\Phi(u,w),v\rangle$$

$\Rightarrow \Phi(u+t_n h,w)\to\Phi(u,w)$. which proves condition (b)

From assumption 2, setting $\eta=\delta_{k-1}u(x)+\delta_{k-1}\varphi(x)$, $\xi=\hat{\delta}_k u(x)+\hat{\delta}_k\varphi(x)$ and $\xi=\hat{\delta}_k w(x)+\hat{\delta}_k\varphi(x)$ and integrating the inequality over Ω:

$$0 \leq \int_\Omega \sum_{|\alpha|\leq k} \Big[a_\alpha(x, \delta_{k-1}u(x) + \delta_{k-1}\varphi(x), \hat{\delta}_k u(x) + \hat{\delta}_k \varphi(x)) - a_\alpha(x, \delta_{k-1}u(x) + \delta_{k-1}\varphi(x), \hat{\delta}_k \omega(x) + \hat{\delta}_k \varphi(x)) \Big] (D^\alpha u - D^\alpha \omega)\, dx$$

$$= \int_\Omega \sum_{1\leq|\alpha|\leq k} a_\alpha(x, \delta_{k-1}u(x) + \delta_{k-1}\varphi(x), \hat{\delta}_k u(x) + \hat{\delta}_k \varphi(x)) D^\alpha(u-\omega)\, dx$$

$$+ \int_\Omega \sum_{1\leq|\alpha|\leq k-1} a_\alpha(x, \delta_k u(x) + \delta_k \varphi) D^\alpha(u-\omega)\, dx - \langle f, v\rangle_Q - \langle g, v\rangle_Q$$

$$- \int_\Omega \sum_{1\leq|\alpha|\leq k} a_\alpha(x, \delta_{k-1}u(x) + \delta_{k-1}\varphi(x), \hat{\delta}_k \omega(x) + \hat{\delta}_k \varphi(x)) D^\alpha(u-\omega)\, dx$$

$$- \int_\Omega \sum_{1\leq|\alpha|\leq k-1} a_\alpha(x, \delta_k u(x) + \delta_k \varphi) D^\alpha(u-\omega)\, dx + \langle f, v\rangle_Q + \langle g, v\rangle_Q$$

$$= \langle \phi(u,u) - \phi(\omega,u), u-\omega \rangle$$

which proves condition (c).

Condition (d) and (e) require additional theorems from Lebesgue integral theory and compact Sobolev embedding - skip here. $\square$