Nonlinear Differential Equations

Practical Exercises 2

Due: 6th March 2024

1. Prove the following theorem:

   Theorem 1.14. *Let X be a finite-dimensional normed linear space and f a continuous mapping defined on a closed, convex, and bounded subset $K \subset X$ mapping K to itself; i.e., $f(x) \in K$ for all $x \in K$. Then, there exists a fixed point of f in K; i.e., $\exists \overline{x} \in K$ such that*

   $$\overline{x} = f(\overline{x}).$$

   Hint. Define

   $$x = \sum_{i=1}^{n} \alpha_i x_i,$$

   where $x_1, \ldots, x_n$ form a basis for X ($\dim X = n$), and $\boldsymbol{\alpha} = \{\alpha_i\}_i^n \in \mathbb{R}^n$. Then, define a linear, continuous operator $T : X \to \mathbb{R}^n$ as $T(x) = \boldsymbol{\alpha}$ (which has a continuous inverse T^{-1}). Defining $K_1 = T(K)$ and $g(\boldsymbol{\alpha}) = T(f(T^{-1}(\boldsymbol{\alpha})))$, show that g has a fixed point, and hence, that f has a fixed point.

2. Prove the following theorem:

   Theorem 1 (Krasnoselskii Fixed Point Theorem)**.** *Let X be a Banach space, and $M \subset X$ be closed, bounded, and convex. Furthermore, consider the mappings $T_1, T_2 : M \to X$ such that*

   - *$T_1(x) + T_2(x) \in M$ for all $x, y \in M$,*
   - *T_1 is strongly contractive, and*
   - *T_2 is continuous and compact.*

   Then, $T_1 + T_2$ has a fixed point in M.

   Hint. Show that

   (a) $I - T_1$ defines a homeomorphism on M, where I is the identity function,
   (b) $T_2 : M \to (I - T_1)(M)$, and
   (c) $(I - T_1)^{-1} \circ T_2 : M \to M$ is compact and continuous (see Lemma 1.17).