1. <u>Theorem 1.14</u> Let X be finite dimensional normed linear space and f a continuous mapping defined on a closed, convex, and bounded subset $K \subset X$ mapping K to itself; i.e $f(x) \in K$ $\forall x \in K$. Then, there exists a fixed point of f in K.

<u>Proof</u>

- Choose basis $x_1, \ldots, x_n$ in X; then, X can be written as $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha = (\alpha_i)_{i=1}^{n} \in \mathbb{R}^n$
- Define $T: X \to \mathbb{R}^n$, $T(x) = \alpha$ - linear, continuous and maps X to $\mathbb{R}^n$. The inverse $T^{-1}: \mathbb{R}^n \to X$ exists and is continuous.

- $K_1 = T(K)$ and define
$$g(\alpha) = T \circ f \circ T^{-1}(\alpha)$$
- K_1 convex, bounded, and closed in $\mathbb{R}^n$, g is continuous and maps K_1 to itself:

<u>K_1 convex</u>: $\forall \alpha = T(x), \beta = T(y) \in K_1, x, y \in K$

As K convex $\lambda x + (1-\lambda) y \in K$, $\lambda \in [0,1]$
$$\Rightarrow T(\lambda x + (1-\lambda) y) \in K_1$$
$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad y = \sum_{i=1}^{n} \beta_i x_i$$
$$\Rightarrow \lambda x + (1-\lambda) y = \sum_{i=1}^{n} (\lambda \alpha_i + (1-\lambda) \beta_i) x_i$$
$$\Rightarrow T(\lambda x + (1-\lambda) y) = (\lambda \alpha_i + (1-\lambda) \beta_i)_{i=1}^{n} \in K_1$$
$$= \lambda \alpha + (1-\lambda) \beta \in K_1$$

<u>K_1 bounded</u> – T homeomorphism from K to K_1

$\Rightarrow$ T closed mapping

$\Rightarrow$ T maps closed set K to closed set K_1

T homeomorphism:

- T & T^{-1} continuous
- $T: K \to K_1 = T(T)$ clearly surjective
- T injective:

Assume $x, y \in K$, $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{i=1}^{n} \beta_i x_i$

If $T(x) = T(y)$

$$\Rightarrow (\alpha_i)_{i=1}^{n} = (\beta_i)_{i=1}^{n} \Rightarrow \alpha_i = \beta_i,\ i = 1, \dots, n$$

$$\Rightarrow x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \beta_i x_i = y$$

Therefore, $T(x) = T(y) \Rightarrow x = y$

<u>g continuous</u> $T, T^{-1}, \& f^{-1}$ all continuous.

<u>g maps K_1 to K_1</u>

$\forall \alpha \in K_1 \quad \exists x \in K$ s.t. $T(x) = \alpha \quad (T^{-1}(\alpha) = x)$

$\forall x \in K \quad \exists y \in K$ s.t. $f(x) = y$

$\forall y \in K \quad \exists \beta \in K_1$ s.t. $T(y) = \beta$

$\Rightarrow \forall \alpha \in K_1 \ \exists \beta \in K_1$ s.t.

$$g(\alpha) = T \circ \underbrace{f \circ \overbrace{T^{-1}(\alpha)}^{x}}_{y} = \beta$$

$\Rightarrow \forall \alpha \in K_1 \ g(\alpha) \in K_1 \Rightarrow g$ maps K_1 to K_1

- g continuous on closed, convex, bounded set $K_1 \subset \mathbb{R}^n$ mapping K_1 to K_1

$$\Rightarrow \exists \bar{\alpha} \text{ s.t. } g(\bar{\alpha}) = \bar{\alpha} \quad (\text{Th } 1.13)$$

$$\Rightarrow \exists \bar{x} \text{ s.t. } T^{-1}(\bar{\alpha}) = \bar{x} \Rightarrow \bar{\alpha} = T(\bar{x})$$

$$g(\bar{\alpha}) = \bar{\alpha} \Rightarrow \quad T \circ f \circ T^{-1}(\bar{\alpha}) = T(\bar{x})$$

$$T \circ f(\bar{x}) = T(\bar{x})$$

$$f(\bar{x}) = \bar{x} \quad \text{as } T \text{ bijective}$$

$\Rightarrow f$ has fixed point $\blacksquare$

3. Theorem (Krasnoselskii) Let X be a Banach space, and $M \subset X$ be closed, bounded, and convex. Furthermore, consider mappings $T_1, T_2 : M \to X$ s.t

- $T_1(x) + T_2(y) \in M \quad \forall x, y \in M$
- T_1 is strongly contractive
- T_2 is continuous and compact

Proof

a) $I - T_1 : M \to (I-T_1)(M)$ homeomorphism on M:

(i) $I - T_1$ bijective - surjective by definition and injective:

Assume not injective; i.e, $\exists x, y \in M, x \neq y$ such that

$$(I-T_1)(x) = (I-T_1)(y)$$
$$x - T_1(x) = y - T_1(y)$$
$$x - y = T_1(x) - T_1(y)$$
$$\Rightarrow \|x-y\| = \|T_1(x) - T_1(y)\| \leq k\|x-y\|, \ k \in (0,1)$$

$\hookrightarrow$ clearly contradiction $\Rightarrow$ injective

(ii) Continuous:

$$\|(I-T_1)(x) - (I-T_1)(y)\| \leq \|x-y\| + \|T_1(x) + T_2(y)\|$$
$$\leq (1+k)\|x-y\|$$

(Lipschitz cont.)

(iii) Inverse Continuous:

$$\|x-y\| \leq \|(I-T_1)(x) - (I-T_1)(y)\| + \|T_1(x) - T_1(y)\|$$
$$\leq \|(I-T_1)(x) - (I-T_1)(y)\| + k\|x-y\|$$
$$\Rightarrow (1-k)\|x-y\| \leq \|(I-T_1)(x) - (I-T_1)(y)\|$$
$$\Rightarrow \|(I-T_1^{-1})(\bar{x}) + (I-T_1)^{-1}(\bar{y})\| \leq \frac{1}{(1-k)}\|\bar{x} + \bar{y}\|$$

where $\bar{x} = (I-T_1)(x)$ & $\bar{y} = (I-T_1)(y)$

b) $T_2 : M \to (I-T_1)(M)$:

For any $x, y \in M$ $T_1(x)+T_2(y) \in M$

Define $F_y : M \to M$ for fixed $y \in M$

$x \mapsto T_1(x)+T_2(y)$

Note that

$$\|F_y(x_1)-F_y(x_2)\| = \|T_1(x_1)-T_1(x_2)\| \leq k\|x_1-x_2\|$$

$\Rightarrow F_y$ strongly contractive

From Banach F.P (Th 1.11) $\Rightarrow \exists! \bar{x} \in M$ s.t.

$$F_y(\bar{x}) = \bar{x}$$

$$T_1(\bar{x})+T_2(y) = \bar{x}$$

$$T_2(y) = \bar{x} - T_1(\bar{x})$$

$$= (I-T_1)(\bar{x}) \in (I-T_1)(M)$$

$\Rightarrow \forall y \in M$ $T_2(y) \in (I-T_1)(M)$

$\Rightarrow T_2 : M \to (I-T_1)(M)$

c) $(I-T_1)^{-1} \circ T_2 : M \to M$ compact & continuous:

T_2 is compact & continuous &

$(I-T_1)^{-1}$ is continuous on Banach spaces

- so apply Lemma 1.17

- So as $M \subset X$ closed, bounded, convex and X a Banach spce, by Th. 1.16 $\exists \xi \in M$ such that

$(I-T_1)^{-1} \circ T_2(\xi) = \xi \Rightarrow T_2(\xi) = (I-T_1)(\xi)$

$\Rightarrow T_2(\xi) = \xi - T_1(\xi) \Rightarrow (T_1+T_2)(\xi) = \xi$

$\Rightarrow \xi$ is a fixed point of T_1+T_2 $\square$