Nonlinear Differential Equations

Practical Exercises 4

Due: 20th March 2024

1. Let $A : X \to X^*$, where X is a real reflexive Banach space. Suppose that the decomposition $A = B + T$ exists, where $B : X \to X^*$ is radially continuous, monotone, and bounded and $T : X \to X^*$ is strongly continuous.

   Show that the operator A is pseudomonotone and bounded.

2. Let X_n be a finite dimensional subspace of a Banach space X. Define $P_n : X \to X_n$ as a *projection* from X to X_n, and construct the dual projection $P_n^* : X_n^* \to X^*$ such that

   $$\langle f_n, P_n x \rangle = \langle P_n^* f_n, x \rangle \qquad \forall f_n \in X_n^*, x \in X.$$

   Given an operator $A : X \to X^*$, define the operator $A_n : X_n \to X_n^*$ as $A_n = P_n^* A P_n$. Prove that

   (a) A monotone $\implies$ A_n monotone
   (b) A coercive $\implies$ A_n coercive
   (c) A demicontinuous $\implies$ A_n demicontinuous

3. Prove the following.

   Theorem 2.12. *Let H be a Hilbert space and the operator $A : H \to H$ meet the following conditions:*

   (a) A is monotone,
   (b) A is continuous for every finite dimensional subspace $M \subset H$,
   (c) A is coercive; i.e.,

   $$\lim_{\|x\| \to \infty} \frac{(Ax, x)}{\|x\|} = \infty.$$

   Then, the operator A is surjective; i.e., the equation $Ax = y$ has a solution for every right hand side $y \in H$. Moreover, if A is strictly monotone the solution is unique.

   Hint. For fixed $y \in H$ define the operator $T : H \to H$ as $Tx = Ax - y$. Then, use the following theorem and/or corollary to show that there exist a x such that $Tx = 0$.

   Theorem 1. *Let B be a closed unit ball in a Hilbert space H, and let $A : B \to H$ be monotone and continuous on $B \cap M$, where M is an arbitrary finite dimensional subspace of H.*

   (a) Then, there exists $x_0 \in B$ such that

   $$(Ax_0, y - x_0) \geq 0 \qquad \textit{for all } y \in B.$$

   Moreover, the set of points satisfying this condition is convex.

   (b) If $\|x_0\| < 1$ then $Ax_0 = 0$.

(c) *If* $\|x_0\| = 1$ *and if, for each*

$$x \in \mathcal{G} := \{u \in H : \|u\| = 1\},$$

it holds that $x + \lambda A x \neq 0$ *for all* $\lambda > 0$, *then* $Ax_0 = 0$.

Corollary 2. *Let* $B(w, r)$ *be a closed ball in a Hilbert space* H *with centre* $w \in H$ *and radius* $r > 0$, *and let* $T : B(w, r) \to H$ *be monotone and continuous on* $B(w, r) \cap M$, *where* M *is an arbitrary finite dimensional subspace of* H. *If, for each*

$$z \in \mathcal{S}(w, r) := \{u \in H : \|u - w\| = r\},$$

it holds that $z - w + \lambda T z \neq 0$ *for all* $\lambda > 0$*; then, there exists a* $z_0 \in B(w, r)$ *such that* $Tz_0 = 0$.

4. Prove the following.

Corollary 2.16. *Let* X *be a separable, reflexive Banach space and the operator* $A : X \to X^*$ *meets one of the following conditions:*

(a) *A is coercive, demicontinuous, bounded, and satisfies condition* (S),

(b) *A is coercive, pseudomonotone, and bounded.*

Then, the operator A *is surjective; i.e, a solution of the equation* $Au = b$ *exists for every right hand side* $b \in X^*$. *Additionally,* A^{-1} *is bounded; i.e., there exists a function* $N : \mathbb{R}_+ \to \mathbb{R}_+$ *such that, for all* $u \in X$,

$$\|u\| \leq N(\|Au\|).$$