1. Let $A: X \to X^*$, where X is real reflexive Banach space. Suppose that decomposition $A = B + T$ exists, where $B: T \to T^*$ is radially continuous, monotone, and bounded and $T: X \to X^*$ is strongly continuous. Show A is pseudomonotone and bounded

A bounded:

$$\|Au\| = \|Bu + Tu\| \leq \|Bu\| + \|Tu\|$$

As B is bounded: $\exists M_B: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Bu\| \leq M_B(\|u\|)$$

As T is strongly continuous $\Rightarrow$ bounded (lemma 2.6)

$\Rightarrow \exists M_T: \mathbb{R}_+ \to \mathbb{R}_+$ s.t. $\|Tu\| \leq M_T(\|u\|)$

Therefore,

$$\|Au\| \leq M_B(\|u\|) + M_T(\|u\|) = M_A(\|u\|)$$

where $M_A: \mathbb{R}_+ \to \mathbb{R}_+$, $M_A(r) := M_B(r) + M_T(r) \Rightarrow A$ bounded

A pseudomonotone:

- T strongly continuous $\Rightarrow$ T pseudomonotone
- B monotone; therefore,

  B radially continuous $\Leftrightarrow$ B hemicontinuous
- B monotone & hemicontinuous $\Rightarrow$ B pseudomonotone

2. Let X_n be finite dimensional subspace of a Banach space X. Define $P_n : X \to X_n$ as a projection from X to X_n and construct dual projection $P_n^* : X_n^* \to X^*$ such that

$$\langle f_n, P_n x\rangle = \langle P_n^* f_n, x\rangle \quad \forall f_n \in X_n^*, x \in X$$

Given an operator $A : X \to X^n$ define $A_n = P_n^* A P_n$.

a) Show A monotone $\Rightarrow A_n$ monotone

For $u_n, v_n \in X_n \subset X$

$$\begin{aligned}\langle A_n u_n - A_n v_n, u_n - v_n\rangle &= \langle P_n^* A P_n u_n, u_n - v_n\rangle - \langle P_n^* A P_n v_n, u_n - v_n\rangle \\ &= \langle A P_n u_n, P_n u_n - P_n v_n\rangle - \langle A P_n v_n, P_n u_n - P_n v_n\rangle \\ &= \langle A u_n - P v_n, u_n - v_n\rangle \quad \text{as } P_n u_n = u_n,\ P_n v_n = v_n\ ! \\ &\geq 0 \quad (\text{as } A \text{ monotone}).\end{aligned}$$

b) Show A coercive $\Rightarrow A_n$ coercive

For $u_n \in X_n \subset X$

$$\begin{aligned}\lim_{\|u_n\| \to \infty} \frac{\langle A_n u_n, u_n\rangle}{\|u_n\|} &= \lim_{\|u_n\| \to \infty} \frac{\langle P_n^* A P_n u_n, u_n\rangle}{\|u_n\|} \\ &= \lim_{\|u_n\| \to \infty} \frac{\langle A P_n u_n, P_n u_n\rangle}{\|u_n\|} \\ &= \lim_{\|u_n\| \to \infty} \frac{\langle A u_n, u_n\rangle}{\|u_n\|} = +\infty\end{aligned}$$

c) A demicontinuous $\Rightarrow A_n$ demicontinuous

$u, u_n \in X_n \quad u_n \to u$

$$\begin{aligned}\forall x \in X_n \quad \langle A_n u_n, x\rangle &= \langle P_n^* A P_n u_n, x\rangle \\ &= \langle A P_n u_n, P_n x\rangle \\ &= \langle A u_n, x\rangle\end{aligned}$$

As A demicontinuous & $u_n \to u \Rightarrow A u_n \rightharpoonup A u$
$\Rightarrow \langle A u_n, x\rangle \to \langle A u, x\rangle$

$$\begin{aligned}\Rightarrow \langle A_n u_n, x\rangle = \langle A u_n, x\rangle \to \langle A u, x\rangle \quad &\forall x \in X_n \\ = \langle A P_n u, P_n x\rangle \quad &\forall x \in X_n \\ = \langle A_n u, x\rangle\end{aligned}$$

$\Rightarrow A_n u_n \rightharpoonup A_n u$

3. Prove Theorem 2.12

For fixed $y \in H$ define $T: H \to H$ as $Tx = Ax - y$, $x \in H$. Then,

a) T is monotone

$$(Tx_1 - Tx_2, x_1 - x_2) = (Ax_1 - Ax_2, x_1 - x_2) \geq 0$$
$$\forall x_1, x_2 \in H$$

b) T is continuous for every finite dimensional subspace $M \subset H$

c) T is coercive

$$\lim_{\|x\| \to \infty} \frac{(Tx, x)}{\|x\|} = \lim_{\|x\| \to \infty} \left(\frac{(Ax, x)}{\|x\|} - \frac{(y, x)}{\|x\|} \right) = \infty$$

Hence, $\exists r > 0$ s.t. for $\|x\| > r$ $(Tx, x) \geq \|x\|$

By contradiction can show that if $\|x\| = r$ that $Tx \neq \delta x$ for all $\delta < 0$:

Let $Tx = \delta x$ for some $\delta < 0$ & $\|x\| = r$;

Then, $\delta \|x\|^2 = (\delta x, x) = (Tx, x) \geq \|x\|$

$$\Rightarrow \delta r^2 \geq r$$

which contradicts that $\delta < 0$.

- From this & definction of T, for $\|x\| = r$ & all $\delta < 0$ $Tx \neq \delta x$. Thus, by Th 1./Cor. 2 $\exists x \in H$ s.t $Tx = 0$

$$\Rightarrow Ax = y.$$

Uniqueness

- Assume $\exists x_1, x_2$ $x_1 \neq x_2$ s.t $Ax_1 = y$, $Ax_2 = y$

Then $(Ax_1 - Ax_2, x_1 - x_2) = (0, x_1 - x_2) = 0$;

however, by strictly monotone $(Ax_1 - Ax_2, x_1 - x_2) > 0$

$\Rightarrow$ contradiction $\Rightarrow x_1 = x_2$

$\Rightarrow$ solution unique

4. Prove Corollary 2.16

a) A coercive, demicontinuous, bounded & satisfies (S)

- From Lemma 2.6

  A demicontinuous $\Rightarrow$ A continuous on FDS

- From Lemma 2.10

  - (S) holds $\Rightarrow$ $(S)_0$ holds
  - demicontinuous & $(S)_0$ holds $\Rightarrow$ $(M)_0$ holds

Therefore, A coercive, cont. on finite dim. subspaces, bounded, and satisfies $(M)_0$

$\Rightarrow$ Conditions of Th. 2.15 met

b) A coercive, pseudomonotone, and bounded

- From Lemma 2.10

  - pseudomonotone $\Rightarrow$ (M) holds $\Rightarrow$ $(M)_0$ holds
  - pseudomonotone & bounded $\Rightarrow$ demicontinuous

- From Lemma 2.6

  demicontinuous $\Rightarrow$ A continuous on FDS

Therefore, A coercive, cont. on finite dim. subspaces, bounded & satisfies $(M)_0$

$\Rightarrow$ conditions of Th. 2.15 met.