Nonlinear Differential Equations

Practical Exercises 6

Due: 3rd April 2024

Exercises

1. State the order of the following differential equations defined on the domain $\Omega = \mathbb{R}^n$, $n = 2, 3$, and categorize them as *linear*, *semilinear*, *quasilinear*, or *(fully) nonlinear*. In the following equations, for a scalar-valued function u, vector-valued function $\boldsymbol{v}$, and $n \times n$ matrix-valued function $\underline{\sigma}$, we define the gradient operator as

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right)^\top \qquad \nabla \boldsymbol{v} := \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{pmatrix},$$

and the divergence operator as

$$\nabla \cdot \boldsymbol{v} := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \qquad \nabla \cdot \underline{\sigma} = \left(\sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x_j}, \ldots, \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x_j}, \right)^\top.$$

For a function ψ we define the Laplacian operator as $\Delta\psi \equiv \nabla \cdot \nabla\psi$, the biharmonic operator as $\Delta^2\psi \equiv \Delta\Delta\psi$, and the Hessian operator as $\nabla^2 = \nabla(\nabla\psi)$.

Poisson's Equation

$$\Delta u = f$$

where $u : \Omega \to \mathbb{R}$ is the unknown and $f : \Omega \to \mathbb{R}$ is a known function.

Chladny sound figures

$$\Delta u + \lambda \sin u = 0$$

where $u : \Omega \to \mathbb{R}$ is the unknown and $\lambda : \Omega \to \mathbb{R}$ is a known function.

von Kármán equations For $n = 2$,

$$D\Delta^2 u - [u, w] = f,$$
$$\Delta^2 w + \frac{1}{2}[u, u] = 0,$$

where $u, w : \Omega \to \mathbb{R}$ are the unknowns, D is a constant, $f : \Omega \to \mathbb{R}$ is a known function, and the bracket operator $[\cdot, \cdot]$ is defined as

$$[u, v] = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}.$$

Non-Newtonian fluid

$$-\nabla \cdot \{\mu(\boldsymbol{x}, |\underline{e}(\boldsymbol{u})|)\, \underline{e}(\boldsymbol{u})\} + \nabla p = \boldsymbol{f}, \quad (1)$$
$$\nabla \cdot \boldsymbol{u} = 0, \quad (2)$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a known source term, $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^n$ is the unknown velocity vector, $p : \mathbb{R}^n \to \mathbb{R}$ is the unknown pressure, $\underline{e}(\boldsymbol{u})$ is the symmetric $n \times n$ strain tensor defined by

$$e_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, \dots, n,$$

$|\underline{e}\,(\boldsymbol{u})|$ is the Frobenius norm of $\underline{e}\,(\boldsymbol{u})$, and $\mu : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a known function.

(Steady State) Navier-Stokes

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\boldsymbol{u}$, p and $\boldsymbol{f}$ are the same as for the non-Newtonian fluid and $\nu : \mathbb{R}^n \to \mathbb{R}$ is a known function. Here, $\boldsymbol{v} \cdot \underline{\sigma} \equiv \underline{\sigma}\boldsymbol{v}$ for a vector $\boldsymbol{v}$ and matrix $\underline{\sigma}$.

Monge-Ampère

$$\det(\nabla^2 u) - f(\boldsymbol{x}, u, \nabla u) = 0,$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is the unknown, and $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a known function.