Nonlinear Differential Equations

Practical Exercises 7

Due: 11th April 2024

1. Prove *only* the boundedness of the Nemyckii operator in the following (ignore continuity).

Theorem 3.11. *Let $k \in \mathbb{N}$, $p \geq 1$, $r \geq 1$, $f(x,\xi) \in \mathrm{CAR}$ be a function defined on $x \in \Omega \subset \mathbb{R}^n$ and $\xi \in \mathbb{R}^\kappa$. Suppose there exists a continuous function $c(t) \geq 0$, defined for $t \geq 0$, and function $g \in L^r(\Omega)$ such that for all $\xi \in \mathbb{R}^\kappa$ and for almost all $x \in \Omega$ it holds that*

$$|f(x,\xi)| \leq c\left(\sum_{|\beta|<k-\frac{n}{p}} |\xi_\beta|\right)\left[g(x) + \sum_{k-\frac{n}{p}\leq|\beta|\leq k} |\xi_\beta|^{\frac{q(\beta)}{r}}\right]$$

where

(a) for $|\beta| > k - \frac{n}{p}$

$$q(\beta) = \frac{np}{n-(k-|\beta|)p},$$

(b) for $|\beta| = k - \frac{n}{p}$

$$q(\beta) \geq 1 \qquad \textit{arbitrary.}$$

Then, for each $u \in W^{k,p}(\Omega)$, $f(x, \delta_k u(x)) \in L^r(\Omega)$ and the Nemyckii operator defined by f ($\mathcal{N}(u)(x) = f(x,u(x)), x \in \Omega$) is continuous and bounded from $W^{k,p}(\Omega)$ to $L^r(\Omega)$.

Hint. Using the inequality

$$(a+b)^r \leq 2^{r-1}(a^r + b^r), \qquad a,b>0, \qquad r \in \mathbb{N},$$

show that

$$|f(x,\delta_k u(x))|^r \leq c_1 \left| c\left(\sum_{|\beta|<k-\frac{n}{p}} |D^\beta u(x)|\right)\right|^r \left(|g(x)|^r + \sum_{k-\frac{n}{p}\leq|\beta|\leq k} |D^\beta u(x)|^{q(\beta)}\right),$$

where c_1 is a constant. c is continuous, Ω is bounded and thus by Theorem 3.8 (iii) there exists a constant $c_2 > 0$ such that for all $x \in \overline{\Omega}$

$$|f(x,\delta_k u(x))|^r \leq c_1 c_2 \left(|g(x)|^r + \sum_{k-\frac{n}{p}\leq|\beta|\leq k} |D^\beta u(x)|^{q(\beta)}\right).$$

Integrate over Ω to complete the proof.

2. Prove the following.

Theorem 3.12. *Let $\mathcal{A}$ be a formal differential operator of order $2k$,*

$$(\mathcal{A}u)(x) = \sum_{|\alpha|\le k} D^\alpha a_\alpha(x, \delta_k u(x))$$

and let the functions $a_\alpha \in \mathrm{CAR}$ satisfy, for almost all $x \in \Omega \subset \mathbb{R}^n$ and all $\xi \in \mathbb{R}^\kappa$ the growth condition

$$|a_\alpha(x,\xi)| \le c_\alpha\left(\sum_{|\beta|<k-\frac{n}{p}} |\xi_\beta|\right)\left[g_\alpha(x) + \sum_{k-\frac{n}{p}\le|\beta|\le k} |\xi_\beta|^{r(\alpha,\beta)}\right]$$

where $p > 1$ and

(a) *$c_\alpha(t)$ is a non-negative continuous function of $t \ge 0$, with c_α constant for $k - \frac{n}{p} < 0$,*

(b) *$g_\alpha \in L^s(\Omega)$, where*

$$s = \begin{cases} \frac{q(\alpha)}{q(\alpha)-1} & \text{if } |\alpha| \ge k - \frac{n}{p}, \\ 1 & \text{if } |\alpha| < k - \frac{n}{p}, \end{cases}$$

(c)

$$r(\alpha,\beta) = \begin{cases} \frac{(q(\alpha)-1)q(\beta)}{q(\alpha)} & \text{if } |\alpha| \ge k - \frac{n}{p}, \\ q(\beta) & \text{if } |\alpha| < k - \frac{n}{p}, \end{cases}$$

(d)

$$q(\nu) = \begin{cases} \frac{Np}{N-(k-|\nu|)p} & \text{if } |\nu| > k - \frac{n}{p}, \\ \ge 1 \text{ arbitrary} & \text{if } |\nu| = k - \frac{n}{p}. \end{cases}$$

Then, the operator A defined by the relation

$$\langle Au, v\rangle = \sum_{|\alpha|\le k} \int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\,\mathrm{d}x, \qquad v \in W^{k,p}(\Omega)$$

is bounded and continuous from $W^{k,p}(\Omega)$ to the dual space $(W^{k,p}(\Omega))^*$.

Hint. To estimate integrals of the form

$$\int_\Omega a_\alpha(x, \delta_k u(x)) D^\alpha v(x)\,\mathrm{d}x, \qquad u, v \in W^{k,p}(\Omega), |\alpha| \le k$$

use Theorems 3.8 and 3.11, taking the functions a_α instead of f and choose r as follows:

$$r = \begin{cases} \frac{q(\alpha)}{q(\alpha)-1}, & \text{if } |\alpha| \ge k - \frac{n}{p}, \\ 1, & \text{if } |\alpha| < k - \frac{n}{p}. \end{cases}$$