1. a) Prove T bounded and demicontinuous

By Theorem 3.12 A bounded from $W^{k,p}(\Omega)$ to $(W^{k,p}(\Omega))^*$.

$$\langle Tu, v\rangle \leq (\|Au\|_{V^*} + \|A\varphi\|_{V^*} + \|f\|_Q + \|g\|_V)\|v\|_V$$
$$\leq (M(\|u\|_V) + M(\|\varphi\|_V) + \|f\|_Q + \|g\|_V)\|v\|_V$$
$$\|Tu\|_{V^*} = \sup_{v\in V} \frac{\langle Tu, v\rangle}{\|v\|_V} \leq M_T(\|u\|_V)$$

where $M_T(\|u\|_V) = M(\|u\|_V) + M(\|\varphi\|_V) + \|f\|_Q + \|g\|_V$.

Assume $u_n \to u$ in V. From continuity of Nemyckii operator (Theorem 3.11) it follows that
$$a_\alpha(x, \delta_k u_n + \delta_k\varphi) \to a_\alpha(x, \delta_k u_0 + \delta_k\varphi)$$
in the space $L^r(\Omega)$, $r \geq 1$, and thus for any $v \in V$ and $\alpha, |\alpha| \leq k$ it holds that
$$\lim_{n\to\infty}\int_\Omega a_\alpha(x, \delta_k u_n + \delta_k\varphi) D^\alpha v\, dx$$
$$= \int_\Omega a_\alpha(x, \delta_k u_0 + \delta_k\varphi) D^\alpha v\, dx$$

Therefore, for any $v \in V$
$$\lim_{n\to\infty}\langle Tu_n, v\rangle = \langle Tu_0, v\rangle;$$
i.e, $Tu_n \to Tu_0$ in $V^* \Rightarrow T$ is demicontinuous.

b) Prove T is monotone

$$\langle Tu - Tv, u - v\rangle = \langle A(u+\varphi) - A(v+\varphi), u - v\rangle$$
$$= \sum_{|\alpha|\leq k}\int_\Omega \underbrace{[a_\alpha(x, \delta_k u + \delta_k\varphi) - a_\alpha(x, \delta_k v + \delta_k\varphi)][D^\alpha u - D^\alpha v]}_{\geq 0 \text{ by condition}} dx$$

≥ 0

c) Prove T is strictly monotone

Identical to above, but (1) holds with ≥ 0 when $u \neq v$ ($\Rightarrow \delta_k u \neq \delta_k v$).

2. a) Define

(i) coefficient functions and:

$$a_{(0,0)}(x,\xi) := b(x, \xi_{(0,0)})$$
$$a_{(1,0)}(x,\xi) := \mu(x,\hat{\xi})\,\xi_{(1,0)}$$
$$a_{(0,1)}(x,\xi) := \mu(x,\hat{\xi})\,\xi_{(0,1)}$$ where $\hat{\xi} = (\xi_{(1,0)}, \xi_{(0,1)})$

(ii) technical already in divergence form!

$$\frac{\partial}{\partial x_1}\left(\mu(x,\nabla u)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\mu(x,\nabla u)\frac{\partial u}{\partial x_2}\right) + b(x,u) = h(x) \text{ in } \Omega \quad (1)$$

(iii) BVP (A, V, Q):

Find solution of (1) satisfying $u = 0$ on $\partial\Omega$

where
$$Au := \frac{\partial}{\partial x_1}\left(\mu(x,\nabla u)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\mu(x,\nabla u)\frac{\partial u}{\partial x_2}\right) + b(x,u)$$
$$V = W_0^{1,2}(\Omega) = H_0^1(\Omega)$$
$$Q = L^2(\Omega)$$

(iv) weak solution of BVP:

Find $u \in W_0^{k,p}(\Omega)$ such that

$$\int_\Omega \mu(x,\nabla u)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \mu(x,\nabla u)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} + b(x,u)v\,dx = \langle f, v\rangle_Q$$

for all $v \in W_0^{k,p}(\Omega)$, where

$$\langle f, v\rangle_Q = \int_\Omega hv\,dx \qquad \forall v \in W_0^{k,p}(\Omega).$$

(v) operator T such that solutions to $Tu = 0$ equivalent to weak solutions of BVP:

$$\langle Tu, v\rangle = \int \mu(x,\nabla u)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \mu(x,\nabla u)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} + b(x,u)v - hv\,dx$$

b) Derive conditions for $\mu, b, \&\ h$ such that Theorem 3.18 can be applied; i.e,

(i) T is monotone:

By Lemma 3.15 require that $\forall \xi, \eta \in \mathbb{R}^3$

$$\sum_{|\alpha|\leq 1} [a_\alpha(x,\xi) - a_\alpha(x,\eta)](\xi_\alpha - \eta_\alpha) \geq 0 \ ; \text{ i.e}$$

$$0 \leq \left(\mu(x,\hat{\xi})\xi_{(1,0)} - \mu(x,\hat{\eta})\eta_{(1,0)}\right)\left(\xi_{(1,0)} - \eta_{(1,0)}\right)$$
$$+ \left(\mu(x,\xi)\xi_{(0,1)} - \mu(x,\eta)\eta_{(0,1)}\right)\left(\xi_{(0,1)} - \eta_{(0,1)}\right)$$
$$+ \left(b(x,\xi_{(0,0)}) - b(x,\eta_{(0,0)})\right)\left(\xi_{(0,0)} - \eta_{(0,0)}\right)$$
$$= \left(\mu(x,\hat{\xi})\hat{\xi} - \mu(x,\hat{\eta})\hat{\eta}\right)\left(\hat{\xi} - \hat{\eta}\right)$$
$$+ \left(b(x,\xi_{(0,0)}) - b(x,\eta_{(0,0)})\right)\left(\xi_{(0,0)} - \eta_{(0,0)}\right)$$

This can be enforced with two conditions:

- $\left(\mu(x,\xi)\xi - \mu(x,\eta)\eta\right)(\xi - \eta) \geq 0 \quad \forall \xi, \eta \in \mathbb{R}^2$
- $\left(b(x,u) - b(x,v)\right)(u - v) \geq 0 \quad \forall u, v \in \mathbb{R}$.

(ii) T is strictly monotone

Require that

$$\left(\mu(x,\hat{\xi})\hat{\xi} - \mu(x,\hat{\eta})\hat{\eta}\right)\left(\hat{\xi} - \hat{\eta}\right)$$
$$+ b(x,\xi_{(0,0)}) - b(x,\eta_{(0,0)})\left(\xi_{(0,0)} - \eta_{(0,0)}\right) > 0 \quad \forall \xi, \eta \in \mathbb{R}^3$$

Need to change two conditions from previous:

- $\left(\mu(x,\xi)\xi - \mu(x,\eta)\eta\right)(\xi - \eta) > 0 \quad \forall \xi, \eta \in \mathbb{R}^2$
- $\left(b(x,u) - b(x,v)\right)(u - v) > 0 \quad \forall u, v \in \mathbb{R}$.

(iii) T is coercive

Need that

$$\sum_{|\alpha|\le 1} \xi_\alpha a_\alpha(x,\xi) \ge C_1 \sum_{|\alpha|\le k} |\xi_\alpha|^p + C_2 |\xi_0|^p - C_3$$

when $p=2$, $C_1>0$ and $C_2, C_3 \ge 0$; i.e.,

$$\mu(x,\hat{\xi})\xi_{(1,0)}^2 + \mu(x,\hat{\xi})\xi_{(1,0)}^2 + b(x,\xi_{(0,0)})$$
$$\ge C_1\left(\xi_{(1,0)}^2 + \xi_{(0,1)}^2\right) + C_2\, \xi_{(0,0)}^2 - C_3.$$

So this is true if $\exists\, C_1>0$ s.t. $\mu(x,\hat{\xi}) \ge C_1$ $\forall \xi \in \mathbb{R}^2$ and $C_2 \ge 0$ such that $b(x,u) \ge C_2 u^2$ $\forall u \in \mathbb{R}$. (and $C_3=0$).

(iv) $a_\alpha \in CAR^*(p)$, $|\alpha|\le 1$

$\Rightarrow b(x,\xi_{(0,0)}) \in CAR$, $\mu(x,\hat{\xi})\xi_{(1,0)} \in CAR$

$\mu(x,\hat{\xi})\xi_{(0,1)} \in CAR$

if $b(x,\cdot)$ and $\mu(x\ \ )$ are continuous almost everywhere for $x\in\Omega$ and $b(\cdot,u)$, $\mu(\cdot,\xi)\xi_1$, $\mu(\cdot,\xi)\xi_2$ are measurable on Ω for fixed $u\in\mathbb{R}$, $\xi\in\mathbb{R}^2$.

Now just need to meet growth condition:

$$|a_\alpha(x,\xi)| \le C_\alpha\left(g_\alpha(x) + \sum_{|\beta|\le 1} |\xi_\beta|^{r(\alpha,\beta)}\right)$$

(for simplicity $r(\alpha,\beta)=2$ is valid for all possible α & β).

So require $\exists\, C_{(0,0)}, C_{(0,1)}, C_{(1,0)}, g_{(0,0)}, g_{(1,0)}$ & $g_{(0,1)}$ s.t.

$$|b(x,\xi_{(0,0)})| \le C_{(0,0)}\left(g_{(0,0)}(x) + |\xi_{(0,0)}|^2 + |\xi_{(1,0)}|^2 + |\xi_{(0,1)}|^2\right)$$

$$|\mu(x,\xi)\xi_{(0,1)}| \le C_{(0,1)}\left(g_{(0,1)}(x) + |\xi_{(0,0)}|^2 + |\xi_{(1,0)}|^2 + |\xi_{(0,1)}|^2\right)$$

$$|\mu(x,\xi)\xi_{(1,0)}| \le C_{(1,0)}\left(g_{(1,0)}(x) + |\xi_{(0,0)}|^2 + |\xi_{(1,0)}|^2 + |\xi_{(0,1)}|^2\right)$$

c) If $b(u) \equiv 0$ state conditions on μ such that T is

(i) strongly monotone: $\exists M > 0$ s.t.

$$(\mu(x,\xi)\xi - \mu(x,\eta)\eta)(\xi - \eta) \geq M|\xi - \eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^2$$

(ii) Lipschitz continuous: $\exists L > 0$ s.t.

$$|\mu(x,\xi)\xi - \mu(x,\eta)\eta| \leq L|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^2$$