Practical 1 – Examples of differential equations

Consider $\Omega \subset \mathbb{R}^n, n=2,3.$

Consider a scalar valued function u, vector-valued function $\underline{v}$ and matrix-valued function $\underline{\underline{\sigma}}$; define gradient operator as

$$\nabla u := \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix} \qquad \nabla \underline{v} := \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{pmatrix}$$

and divergence operator as

$$\nabla \cdot v := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \qquad \nabla \cdot \underline{\underline{\sigma}} := \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x_j} \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x_j} \end{pmatrix}$$

For function ψ define Laplacian operator as $\Delta \psi \equiv \nabla \cdot \nabla \psi$, biharmonic operator as $\Delta^2 \psi \equiv \Delta \Delta \psi$ and the Hessian operator as $\nabla^2 \psi \equiv \nabla(\nabla \psi)$.

Example 1.2 For unknown $u : \Omega \to \mathbb{R}$ and known $f : \Omega \to \mathbb{R}$

$$-\Delta u(x) = f(x) \text{ in } \Omega$$

– Is this linear, semilinear, quasilinear, or fully nonlinear?

Linear, why? $-\Delta u + f = 0$

$$-\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + f = 0$$

– Order $k=2$, all derivatives of order ≤ 2 are linear

Example 1.3 Assume thin flexible square plate, horizontally fixed at centre. Uniformly distribute tiny particles (e.g. sand) on plate. Above plate fix sound wave source with variable frequency λ. Pressure of sound waves induces load on plate surface. In case of resonance vibration with unmoved nodal lines is induced, with the sand collecting at the nodal lines.

Let $\Omega = [0,L]^2$ and $u(\underline{x}): \Omega \to \mathbb{R}$ denote maximal deviation of vibration from flat position; then, u satisfies (strongly simplified) model:

$$\Delta u + \lambda \sin u = 0 \text{ in } \Omega = [0,L]^2$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

Is this linear, semilinear, quasilinear or (fully) nonlinear?

Semilinear, why $\Delta u + \lambda \sin u = 0$

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda \sin u = 0$$

- Order $k=2$, linear in derivatives of order 2 (and 1)
- Nonlinear in derivative of order 0

Example 1.4 (von Kármán equations). T

Consider a plate P under compression ($\Omega \subset \mathbb{R}^2$). It is defined by Airy stress function $w(\underline{x})$ of P at $\underline{x}$ and deviation/deflection $u(\underline{x})$ for P from forward flat state. This can be modelled by the von Kármán equations:

$$D\Delta^2 u - [u,w] = f$$
$$\Delta^2 w + \frac{1}{2}[u,u] = 0$$

where D is a constant, $f: \Omega \to \mathbb{R}$ known function and $[u,v] := \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}$

- This is semilinear.
- Order $k=4$
  - Linear in fourth order terms
  - Nonlinear / coefficient with second order terms for second order terms.

Example 1.5 (Non-Newtonian fluid).

Non-Newtonian fluids are fluids which do not follow Newton's law of viscosity; i.e, they have variable viscosity dependent on stress. In particular, viscosity of non-Newtonian fluids can change when subject to force; e.g. Ketchup becomes runnier when shaken.

Simplied, steady state, model:

$$-\nabla\cdot\{\mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|)\,\underline{\underline{e}}(\underline{u})\} + \nabla p = \underline{f}$$
$$\nabla\cdot\underline{u} = 0$$

where $\underline{f}:\mathbb{R}^n\to\mathbb{R}^n$ known source term, $\underline{u}:\mathbb{R}^n\to\mathbb{R}^n$ is unknown velocity vector, $p:\mathbb{R}^n\to\mathbb{R}$ is unknown pressure, $\underline{\underline{e}}(\underline{u})$ is the symmetric strain tensor

$$e_{ij}(\underline{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i,j=1,\dots,n$$
$$\underline{\underline{e}}(\underline{u}) := \frac{1}{2}\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)$$

and $|\underline{\underline{e}}(\underline{u})|$ is the Frobenius norm of $\underline{\underline{e}}(\underline{u})$ and μ is a known, potentially nonlinear, function.

This is quasilinear:

$$\nabla \cdot \{ \mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \underline{\underline{e}}(\underline{u}) \}$$

$$= \begin{pmatrix} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \underline{\underline{e}}_{1k}(\underline{u}) \\ \vdots \\ \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \underline{\underline{e}}_{nk}(\underline{u}) \end{pmatrix}$$

$$\frac{\partial}{\partial x_k} \mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \underline{\underline{e}}_{1i}(\underline{u})$$

$$= \mu'(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \frac{\partial}{\partial x_i} |\underline{\underline{e}}(\underline{u})| \; \underline{\underline{e}}_{1k}(\underline{u})$$

$$+ \mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \frac{1}{2} \left(\frac{\partial^2 u_1}{\partial x_k^2} + \frac{\partial^2 u_i}{\partial x_1 \partial x_k} \right)$$

$$\left(\begin{aligned} \frac{\partial}{\partial x_k} |\underline{\underline{e}}(\underline{u})| &= \frac{\partial}{\partial x_k} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{4} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right)^{1/2} \\ &= \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{4} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \right)^{-1/2} \\ &\quad \times \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial^2 u_i}{\partial x_k \partial x_j} + \frac{\partial^2 u_j}{\partial x_k \partial x_i} \right) \end{aligned} \right)$$

$$= \overbrace{\mu'(\underline{x}, |\underline{\underline{e}}(\underline{u})|) \frac{1}{2} |\underline{\underline{e}}(\underline{u})|^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}^{\text{first order deriv}} \underbrace{\left(\frac{\partial^2 u_i}{\partial x_k \partial x_j} + \frac{\partial^2 u_j}{\partial x_k \partial x_i} \right)}_{\text{second order deriv}}$$

$$+ \underbrace{\mu(\underline{x}, |\underline{\underline{e}}(\underline{u})|)}_{\text{first order deriv}} \frac{1}{2} \underbrace{\left(\frac{\partial^2 u_1}{\partial x_k^2} + \frac{\partial^2 u_i}{\partial x_1 \partial x_k} \right)}_{\text{second order derivs}}$$

$\rightarrow$ Linear in second order ($k=2$) terms, but has coefficients dependent on first order terms.

Example 1.6 (Steady-state Navier–Stokes)

$$-\nu \Delta \underline{u} + \underline{u} \cdot \nabla \underline{u} + \nabla p = \underline{f}$$
$$\nabla \cdot \underline{u} = 0$$

where $\nu : \mathbb{R}^n \to \mathbb{R}$ is a known function. Here, $\underline{v} \cdot \underline{\underline{\sigma}} \equiv \underline{\underline{\sigma}} \, \underline{v}$ for vector $\underline{v}$ & matrix $\underline{\underline{\sigma}}$.

$\to$ Semilinear as second order terms $(k=2)$ are linear with coefficients only dependent on $\underline{x}$ but first order terms have coefficient dependin on zero order terms.

Example 1.7 (Monge–Ampère) Consider $n=2$ only.

$$\det(\nabla^2 u) - f(\underline{x}, u, \nabla u) = 0$$

where $u : \mathbb{R}^n \to \mathbb{R}$ unknown, and $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ known.

$\to$ Fully nonlinear:

$$\det(\nabla^2 u) = \det\left(\nabla\left(\nabla u\right)\right)$$
$$= \det\left(\nabla \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}\right)$$
$$= \begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{vmatrix}$$
$$= \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

$\to$ nonlinear in second order terms and $k=2$.