Nonlinear Differential Equations

Practical 3: Banach & Sobolev Spaces

1. Prove the *generalised Hölder's inequality:* For $m \in \mathbb{N}$, $m \geq 2$, let there exists functions f_i, $i = 1, \dots, m$ and $0 \leq p_1, \dots, p_m \leq \infty$; then,

$$\|f_1 \cdots f_m\|_{0,r} \leq \|f_1\|_{0,p_1} \cdots \|f_m\|_{0,p_m} \qquad \text{for } \frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}.$$

Hint. *Use the standard Hölder's inequality (Lemma 1.12) and induction.*

Solution: For $r = \infty$ then $p_1 = \cdots = p_m = \infty$ and

$$\|f_1 \cdots f_m\|_{0,\infty} \leq \|f_1\|_{0,\infty} \cdots \|f_1\|_{0,\infty}$$

follows trivially from properties of the essential supremum.

For $1 \leq r < \infty$ we proceed by induction on m.

Base case: We first consider $m = 2$: We have that ${}^1\!/_r = {}^1\!/_{p_1} + {}^1\!/_{p_2}$; hence, ${}^r\!/_{p_1} + {}^r\!/_{p_2} = 1$ and $1 \leq {}^{p_1}\!/_r + {}^{p_1}\!/_q \leq \infty$. Therefore, from Lemma 1.12,

$$\|f_1 f_2\|_{0,r} = \||f_1|^r |f_2|^r\|_{0,1}^{1/r} \leq \||f_1|^r\|_{0,p_1/r}^{1/r} \||f_2|^r\|_{0,p_2/r}^{1/r} = \|f_1\|_{0,p_1} \|f_2\|_{0,p_2}. \tag{1.1}$$

Induction step: We now assume that the theorem holds for all $m \leq k$ and show it holds for $k+1$. Setting ${}^1\!/_p = \sum_{i=1}^{k} {}^1\!/_{p_i} = {}^1\!/_r - {}^1\!/_{p_{k+1}}$ we have that ${}^1\!/_r = {}^1\!/_p + {}^1\!/_{p_{k+1}}$, with $1 \leq p, q \leq 1$; hence, by (1.1)

$$\|f_1 \cdots f_k f_{k+1}\|_{0,r} \leq \|f_1 \cdots f_k\|_{0,p} \|f_{k+1}\|_{0,p_{k+1}}. \tag{1.2}$$

If $p < \infty$ we can apply the induction hypothesis to get that

$$\|f_1 \cdots f_k\|_{0,p} \leq \|f_1\|_{0,p_1} \cdots \|f_k\|_{0,p_k} \tag{1.3}$$

as ${}^1\!/_p = \sum_{i=1}^{k} {}^1\!/_{p_i}$; or trivially for $p = \infty$. Combining (1.2) and (1.3) completes the proof.

2. Let $\Omega \subset \mathbb{R}^2$ be a measurable domain with Lipschitz boundary and $\alpha \in \mathbb{N}_0^n$ be a multi-index; then, prove that the seminorm

$$|v|_{1,2,\Omega} = \left(\sum_{|\alpha|=1} \|\partial^\alpha v\|_{0,2,\Omega}^2 \right)^{1/2}$$

is a norm on the space $H_0^1(\Omega)$.

Solution: As $|\cdot|_{1,2,\Omega}$ is a seminorm the only property of a norm we need to show is that

$$|v|_{1,2,\Omega} = 0 \quad \iff \quad v = 0.$$

We first note we can re-write the seminorm as

$$|v|_{1,2,\Omega} = \left(\sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{0,2,\Omega}^2 \right)^{1/2}$$

$\Longleftarrow$ If $v = 0$ then $\partial v/\partial x_i = 0$, $i = 1, \ldots, n$; hence,

$$\left\| \frac{\partial v}{\partial x_i} \right\|_{0,2,\Omega} = 0, i = 1, \ldots, n \qquad \Longrightarrow \qquad |v|_{1,2,\Omega} = 0.$$

$\Longrightarrow$ If $|v|_{1,2,\Omega} = 0$ then

$$\sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{0,2,\Omega}^2 = 0 \qquad \Longrightarrow \qquad \left\| \frac{\partial v}{\partial x_i} \right\|_{0,2,\Omega} = 0, i = 1, \ldots, n.$$

By the fact that $\|\cdot\|_{0,2,\Omega}$ is a norm, we have that

$$\frac{\partial v}{\partial x_i} = 0, i = 1, \ldots, n \qquad \Longrightarrow \qquad v = c$$

were $c \in \mathbb{R}$ is a constant. Additionally, as $v \in H_0^1(\Omega)$ then $v = 0$ on the boundary $\partial\Omega$. It can then be shown that this is only valid for $v = c = 0$.

3. Let $F : C^2([0, L]) \to C([0, L])$ be defined by

$$F(\varphi) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \lambda \sin\varphi$$

for fixed $\lambda \in \mathbb{R}$; cf. Example 1.1. Derive the Fréchet derivative in φ and Gâteaux derivative in φ in the direction ψ of F.

Hint. *Consider $F(\varphi+\psi) - F(\varphi)$ and $F(\varphi+t\psi) - F(\varphi)$, respectively, for $\varphi, \psi \in C^2([0, L])$ with small $\|\varphi\|_{2,\infty}$ and $\|\psi\|_{2,\infty}$.*

Solution: We start with the Fréchet derivative.

$$F(\varphi + \psi) - F(\varphi) = \frac{\mathrm{d}^2(\varphi + \psi)}{\mathrm{d}s^2} - \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \lambda(\sin(\varphi + \psi) - \sin\varphi).$$

By Taylor's expansion of $\sin$ around φ:

$$\sin(\varphi + \psi) = \sin(\varphi) + \cos(\varphi)(\varphi + \psi - \varphi) + o(\psi)$$

Therefore,

$$F(\varphi + \psi) - F(\varphi) = \frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + \lambda\cos(\varphi)\psi + o(\psi) = F_F'(\varphi)\psi$$

where $F_F'(\varphi) \in \mathcal{L}(C^2([0,L]), C([0,L]))$ defined as

$$F_F'(\varphi) : \psi \mapsto \frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + \lambda\cos(\varphi)\psi$$

is the Fréchet derivative.

Similarly, we have that

$$\begin{aligned} F(\varphi + t\psi) - F(\varphi) &= \frac{\mathrm{d}^2(\varphi + t\psi)}{\mathrm{d}s^2} - \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \lambda(\sin(\varphi + t\psi) - \sin\varphi) \\ &= t\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + t\lambda\cos(\varphi)\psi + o(t\psi) \\ &= tF_G'(\varphi,\psi) + o_\psi(t) \end{aligned}$$

where $o_\psi(t) = o(t)$ is dependent on ψ and $F_F'(\varphi) \in C([0,L])$ defined as

$$F_G'(\varphi,\psi) = \frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + \lambda\cos(\varphi)\psi$$

is the Gâteaux derivative in φ in the direction ψ

4. Let $\Omega \subset \mathbb{R}^n$ be a measurable domain with Lipschitz boundary and $X = H_0^1(\Omega)$; then, define $F : X \to X'$ be defined such that for $u, v \in X$

$$\langle F(u), v\rangle = \int_\Omega \mu(|\nabla u|)\nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x},$$

where $\mu(t) \in C([0,\infty))$ is the *Carreau law* defined by

$$\mu(t) \coloneqq \mu_{\text{inf}} + (\mu_0 - \mu_{\text{inf}})\left(1 + (\lambda t)^2\right)^{\frac{n-1}{2}}$$

for constants $\mu_{\text{inf}}, \mu_0, n, \lambda \in \mathbb{R}$. Compute

$$\langle F_G'(u,w), v\rangle$$

where $F_G'(u,w)$ is the Gâteaux derivative of F in u in the direction w.

Hint. *Use*

$$\langle F_G'(u,w), v\rangle = \lim_{t\to 0}\frac{\langle F(u+tw) - F(u), v\rangle}{t}.$$

Solution: *Note that above definition of μ is potentially problematic - we proceed as if we can make some assumptions on μ and ignore its actual definition.*

$$\begin{aligned} \langle F_G'(u,w), v\rangle &= \lim_{t\to 0}\frac{\langle F(u+tw) - F(u), v\rangle}{t} \\ &= \lim_{t\to 0}\frac{1}{t}\left(\int_\Omega \mu(|\nabla u + t\nabla w|)\nabla(u+tw)\cdot\nabla v\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mu(|\nabla u|)\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x}\right) \\ &= \lim_{t\to 0}\frac{1}{t}\int_\Omega \left(\mu(|\nabla u + t\nabla w|) - \mu(|\nabla u|)\right)\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} \\ &\quad + \lim_{t\to 0}\int_\Omega \mu(|\nabla u + t\nabla w|)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

$$
\begin{aligned}
\langle F_G'(u,w),v\rangle &= \int_\Omega \lim_{t\to 0}\frac{\mu(|\nabla u + t\nabla w|)-\mu(|\nabla u|)}{t}\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu(|\nabla u|)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \left(\frac{\mathrm{d}}{\mathrm{d}t}\mu(|\nabla u + t\nabla w|)\bigg|_{t=0}\right)\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu(|\nabla u|)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x}
\end{aligned}
$$

Then, assuming that the derivative of $\mu(t)$, with respect to the argument, exists and is defined by $\mu'(t)$; then,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mu(|\nabla u + t\nabla w|) \\
&\quad = \mu'(|\nabla u + t\nabla w|)\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}(u+tw)\right)^2\right)^{1/2} \\
&\quad = \mu'(|\nabla u + t\nabla w|)\frac{1}{2}\left(\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}(u+tw)\right)^2\right)^{-1/2}\sum_{i=1}^n\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial x_i}(u+tw)\right)^2 \\
&\quad = \mu'(|\nabla u + t\nabla w|)\left(\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}(u+tw)\right)^2\right)^{-1/2}\sum_{i=1}^n\left(\frac{\partial}{\partial x_i}(u+tw)\right)\frac{\partial w}{\partial x_i}.
\end{aligned}
$$

Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mu(|\nabla u + t\nabla w|)\bigg|_{t=0} = \mu'(|\nabla u|)|\nabla u|^{-1}\left(\nabla u\cdot\nabla w\right).
$$

Combining the above results we get that

$$
\langle F_G'(u,w),v\rangle = \int_\Omega \mu'(|\nabla u|)|\nabla u|^{-1}\left(\nabla u\cdot\nabla w\right)\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu(|\nabla u|)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x}.
$$