# Nonlinear Differential Equations

## Practical 4: Monotone & Continuous Operators

1. Let $X$ be a Banach space, and $A : X \to X'$ be a nonlinear operator. Then prove the following:
   (a) $A$ strongly monotone $\implies$ $A$ uniformly monotone
   (b) $A$ uniformly monotone $\implies$ $A$ strictly monotone
   (c) $A$ strictly monotone $\implies$ $A$ monotone
   (d) $A$ uniformly monotone $\implies$ $A$ (nonlinear) coercive
   (e) $A$ uniformly monotone $\implies$ $A$ stable
   (f) $A$ Lipschitz continuous $\implies$ $A$ continuous
   (g) $A$ strongly continuous $\implies$ $A$ continuous
   (h) $A$ strongly continuous $\implies$ $A$ weakly continuous
   (i) $A$ weakly continuous $\implies$ $A$ demicontinuous
   (j) $A$ continuous $\implies$ $A$ demicontinuous
   (k) $A$ demicontinuous $\implies$ $A$ hemicontinuous

2. Let $X$ be a Banach space, and $A, B : X \to X'$ be nonlinear operators. Then prove the following:
   (a) $A$ strongly monotone and $B$ strongly monotone $\implies$ $A + B$ strongly monotone
   (b) $A$ strongly monotone and $B$ monotone $\implies$ $A + B$ strongly monotone

3. Let $\boldsymbol{A} : \mathbb{R}^m \to \mathbb{R}^m$, $m > 0$, be a symmetric positive definite matrix. Show that the operator $A : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$\langle Au, v\rangle = (\boldsymbol{A}u) \cdot v \qquad \text{for all } v \in \mathbb{R}^m$$

   is strongly monotone and Lipschitz continuous.

   **Hint.** *Consider the eigendecomposition of $\boldsymbol{A}$.*

4. Let $X$ be a Hilbert space, $A : X \to X'$ be strongly monotone and Lipschitz continuous, $f \in X'$ and $J_X$ be the Riesz-isomorphism on $X$.
   (a) Show that there exists a constant $\varepsilon$ such that the mapping $T : X \to X$ defined as

$$T(u) = u - \varepsilon J_X^{-1}(Au - f)$$

   is strongly contractive; i.e,

$$\|T(x) - T(y)\| \leq k\|x - y\| \qquad \text{for all } x, y \in X$$

   with $k^2 = 1 + \varepsilon^2 L^2 - 2\varepsilon M$. Additionally, specify the condition on $\varepsilon$ such that $k \in (0, 1)$.

(b) Compute the optimal value of $\varepsilon$ such that the iteration

$$u_{m+1} = u_m - \varepsilon J_X^{-1}(Au_m - f)$$

converges fastest to the *unique* solution of $Au = f$ and the compute the contraction constant $k$

**Hint.** *From Corollary 2.9 and Banach's fixed point theorem the error is given by*

$$\|u - u_m\| \leq \frac{k^m}{1-k}\|x_0 - x_1\|;$$

*hence, the fastest convergence rate is given when $k$ is close to zero.*