Nonlinear Differential Equations

Practical 7: Semimonotone Operators

Note: This exercise was uploaded late; therefore, do not attempt to complete the exercise this week. Instead, just consider *ideas* on *how* each step can be shown and we will discuss in the practical.

Let X be a real, separable, reflexive Banach space and $B : X \times X :\to X'$ be a map such that

$$Au = B(u, u) \qquad \text{for all } u \in X.$$

The operator $A : X \to X'$ is called *semimonotone* if and only if the following hold.

a) For all $u, v \in X$
$$\langle B(u, u) - B(u, v), u - v\rangle \geq 0.$$

b) For each $u \in X$, the operator $v \mapsto B(u, v)$ is hemicontinuous and bounded from X to X', and, for each $v \in X$, the operator $u \mapsto B(u, v)$ is hemicontinuous and bounded from X to X'.

c) If $u_n \rightharpoonup u$ in X and
$$\lim_{n\to\infty} \langle B(u_n, u_n) - B(u_n, u), u_n - u\rangle = 0;$$
then, $B(u_n, v) \rightharpoonup B(u, v)$ in X' for all $v \in X$,

d) Let $v \in X$, $u_n \rightharpoonup u$ in X, and $B(u_n, v) \rightharpoonup w$ in X' as $n \to \infty$; then,
$$\lim_{n\to\infty} \langle B(u_n, v), u_n\rangle = \langle w, u\rangle.$$

e) A is bounded.

Exercises

1. Let $A : X \to X'$ be a semimonotone operator on a real, separable, reflexive Banach space X, and $B : X \times X \to X'$ the associated map. Assume that $u_n \rightharpoonup u$, $B(u_n, u) \rightharpoonup w$ and
$$\limsup_{n\to\infty} \langle B(u_n, u_n), u_n - u\rangle \leq 0.$$

   (a) Show that
   $$\lim_{n\to\infty} \langle B(u_n, u_n) - B(u_n, u), u_n - u\rangle = 0;$$
   i.e, show that the condition of property c) of a semimonotone operator is satisfied.

   (b) Hence, show that
   $$\langle B(u, u), u - w\rangle \leq \liminf_{n\to\infty} \langle B(u_n, u_n), u_n - u\rangle \qquad \text{for all } x \in X;$$
   i.e., A is a pseudo-monotone operator.

   Hint. *Similar to question 3 from last week.*

2. Consider a quasilinear PDE of order $2k$, $k \in \mathbb{N}$ of the form

$$\sum_{|\alpha|\le k} (-1)^\alpha \partial^\alpha a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x})) = f(\boldsymbol{x}) \qquad \text{in } \Omega,$$

$$\frac{\partial^i u}{\partial \boldsymbol{n}^i} = 0 \qquad \text{on } \partial\Omega, i = 1, \dots, k-1,$$

where Ω is a bounded Lipschitz domain. Let $a_\alpha : \Omega \times \mathbb{R}^\kappa \to \mathbb{R}$, for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$, satisfies the Carathéodory condition **(B1)**, growth condition **(B2)**, and coercivity condition **(C2)** from Theorem 2.19, as well as the following:

(I1) The highest order terms are *strictly monotone* with respect to the highest order derivatives; i.e.,

$$\sum_{|\alpha|=k} \left(a_\alpha(\boldsymbol{x}, \eta, \xi) - a_\alpha(\boldsymbol{x}, \eta, \widehat{\xi}) \right) (\xi_\alpha - \widehat{\xi}_\alpha) > 0,$$

for all $\eta \in \mathbb{R}^{\widetilde{\kappa}}$, $\xi, \widehat{\xi} \in \mathbb{R}^{\widetilde{\kappa}-\kappa}$, where

$$\widetilde{\kappa} = \frac{(n+k-1)!}{n!(k-1)!}$$

is the number of multi-indices of length $|\alpha| \le k-1$.

(I2) The highest order terms are *coercive* with respect to the highest order derivatives; i.e.,

$$\lim_{|\xi|\to\infty} \sup_{\eta\in D} \sum_{|\alpha|=k} \frac{a_\alpha(\boldsymbol{x}, \eta, \xi)}{|\xi| + |\xi|^{p-1}} == \infty,$$

for almost all $x \in \Omega$ and bounded sets $D \subset \mathbb{R}^{\widetilde{\kappa}}$.

Let $A : W_0^{k,p}(\Omega) \to W^{-k,q}(\Omega)$, $Au = B(u,u)$, where

$$\langle B(w,u), v\rangle = \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1} w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x})) \partial^\alpha v \, \mathrm{d}\boldsymbol{x} + \int_\Omega \sum_{|\alpha|\le k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x})) \partial^\alpha v \, \mathrm{d}\boldsymbol{x}.$$

(a) Show that for all $u, v \in W_0^{k,p}(\Omega)$

$$\langle B(u,u) - B(u,v), u - v\rangle \ge 0;$$

i.e., prove property a) of a semimonotone operator.

(b) Show that for each $u \in W_0^{k,p}(\Omega)$, the operator $v \mapsto B(u,v)$ is hemicontinuous and bounded from $W_0^{k,p}(\Omega)$ to $W^{-k,q}(\Omega)$; i.e., prove the first part of property b) of a semimonotone operator.

(c) Show that for each $v \in W_0^{k,p}(\Omega)$, the operator $u \mapsto B(u,v)$ is hemicontinuous and bounded from $W_0^{k,p}(\Omega)$ to $W^{-k,q}(\Omega)$; i.e., prove the second part of property b) of a semimonotone operator.

(d) Show that A is bounded; i.e., prove property e) of a semimonotone operator.

(e) Show that A is coercive.

(f) Assume properties c) and d) of a semimonotone operator applies for A without proof. Hence, show that a solution $u \in W_0^{k,p}(\Omega)$ of the weak formulation

$$a(u,v) := \int_\Omega \sum_{|\alpha|\le k} a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x})) \partial^\alpha v \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v \, \mathrm{d}\boldsymbol{x}, \qquad \text{for all } v \in W_0^{k,p}(\Omega),$$

exists for each right-hand side $f \in L^q(\Omega)$, $1/p + 1/q = 1$.