18.12.2023 — Homework 4

Finite Element Methods 1

Due date: 8th January 2024

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 8th January 2024.

1. (2 points) Let T be an n-simplex in $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_{n+1}$ be the barycentric coordinates with respect to the vertices of T. Prove the formula

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \, \mathrm{d}\boldsymbol{x} = \frac{\alpha_1! \alpha_2! \cdots \alpha_{n+1}! n!}{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} + n)!} |T|, \quad \forall \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{N}_0.$$

   Hint. Transform the integral over T to an integral over the reference simplex $\widehat{T}$.

2. (2 points) Prove Theorem 12 from the lecture:

   Let $\{\mathcal{T}_h\}$ be a family of triangulations of a bounded domain Ω with Lipschitz-continuous boundary and $\{X_h\}$ the family of the corresponding finite element spaces under the assumptions required for convergence of a discrete solution. We assume all finite elements (T, P_T, Σ_T) are affine-equivalent to a reference element $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ with invertible affine mapping $F_T(\widehat{x}) = \mathbb{B}_T \widehat{x} + b_T$ such that $F_T(\widehat{T}) = T$.

   Consider the quadrature formula

$$\int_{\widehat{T}} \widehat{\varphi} \, \mathrm{d}\boldsymbol{x} \approx \sum_{\ell=1}^{L} \widehat{\omega}_\ell \widehat{\varphi}(\widehat{b}_\ell)$$

   where $\widehat{\varphi}$ and $\widehat{b}_\ell$, for $\ell = 1, \ldots, L$, are the quadrature weights and nodes, defined on the reference element $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$, let $m \in \mathbb{N}$ be such that $\widehat{P} \subset P_m(\widehat{T})$, and let there hold at least one of the following conditions

   (I) $\bigcup_{\ell=1}^{L} \{\widehat{b}_\ell\}$ contains a $P_{m-1}(\widehat{T})$-unisolvent set, or

   (II) $\widehat{E}(\widehat{\varphi}) = 0$ for all $\widehat{\varphi} \in P_{2m-2}(\widehat{T})$,

   where

$$\widehat{E}(\widehat{\varphi}) = \int_{\widehat{T}} \widehat{\varphi} \, \mathrm{d}\boldsymbol{x} - \sum_{\ell=1}^{L} \widehat{\omega}_\ell \widehat{\varphi}(\widehat{b}_\ell).$$

Define $V_h = \{v_h \in X_h : \phi(v_h) = 0 \quad \forall \phi \in \Sigma_h^{\partial\Omega}\} \subset H_0^1(\Omega) \cap C(\overline{\Omega})$ and the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ as

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right) (b_{\ell_T}),$$

where $\omega_{\ell,T} = |\det \mathbb{B}_T| \widehat{\omega}_\ell$, $b_{\ell,T} = F_T(\widehat{b}_\ell)$, $a_{ij} \in L^\infty(\Omega)$ and let there exist a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \qquad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n.$$

Prove that the bilinear form a_h is *uniformly* V_h*-elliptic*; i.e., there exists a constant $\widetilde{\alpha} > 0$ independent of h such that

$$a_h(v_h, v_h) \geq \widetilde{\alpha} |v_h|_{1,h}^2 \qquad \forall v_h \in V_h, \forall h > 0.$$

Hint. First show that there exists a constant $\widehat{C} > 0$ such that

$$\widehat{C} |\widehat{p}|_{1,\widehat{T}} \leq \sum_{\ell=1}^{L} \widehat{\omega}_\ell |\widehat{\nabla} \widehat{p}(\widehat{b}_\ell)|^2 \qquad \forall \widehat{p} \in \widehat{P}$$

for condition (I) and (II) separately.

3. (2 points) Let $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ be a finite element, and let $l, m \in \mathbb{N}_0$ and $r, q \in [1, \infty]$ be such that $l \leq m$ and $\widehat{P} \subset W^{l,r}(\widehat{T}) \cap W^{m,q}(\widehat{T})$. For any $T \in \mathcal{T}_h$, let (T, P_T, Σ_T) be a finite element which is affine-equivalent to $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$. Then, there exists a positive constant C, depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, and n such that

$$|v|_{m,q,T} \leq C \frac{h_T^l}{\varrho_T^m} |T|^{\frac{1}{q} - \frac{1}{r}} |v|_{l,r,T}, \qquad \text{for all } v \in P_T, T \in \mathcal{T}_h. \tag{3.1}$$

Let X_h be the finite element space corresponding to $\mathcal{T}_h$ and the finite elements (T, P_T, Σ_T). Introduce the seminorms

$$|v|_{m,q,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,q,T}^q \right)^{1/q} \quad \text{if } q < \infty, \qquad |v|_{m,\infty,h} = \max_{T \in \mathcal{T}_h} |v|_{m,\infty,T}.$$

Let $\mathcal{T}_h$ satisfy

$$\frac{h_T}{\varrho_T} \leq \sigma, \qquad \text{for all } T \in \mathcal{T}_h,$$

and the *inverse assumption*

$$\exists \kappa > 0 : \qquad \frac{h}{h_T} \leq \kappa \quad \text{for all } T \in \mathcal{T}_h,$$

where $h = \max_{T \in \mathcal{T}_h} h_T$. Prove that the *inverse inequality*

$$|v_h|_{m,q,h} \leq C h^{l-m+\min(0,n/q-n/r)} |v_h|_{l,r,h}, \qquad \text{for all } v_h \in X_h,$$

where C is a positive constant depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, κ, and Ω.

Hint. The following inequalities may be useful.

Hölder Inequality For any non-negative numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

for any $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$.

Jensen Inequality For any non-negative numbers $a_1, \dots, a_n$

$$\left(\sum_{i=1}^{n} a_i^q\right)^{1/q} \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}$$

for any $p, q \in (0, \infty)$ satisfying $p \le q$.