18.12.2023 — Homework 4

Finite Element Methods 1

Due date: 8th January 2024

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on the 8th January 2024.

1. (2 points) Let T be an n-simplex in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_{n+1}$ be the barycentric coordinates with respect to the vertices of T. Prove the formula

$$\int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \,\mathrm{d}\boldsymbol{x} = \frac{\alpha_1! \alpha_2! \cdots \alpha_{n+1}! n!}{(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} + n)!} |T|, \quad \forall \alpha_1, \dots, \alpha_{n+1} \in \mathbb{N}_0.$$

Hint. Transform the integral over T to an integral over the reference simplex $\widehat{T}$.

Solution:

Let F_T be an invertible mapping which maps the unit n-simplex $\widehat{T}$ onto the n-simplex T. Then,

$$\begin{aligned} I &\coloneqq \int_T \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{n+1}^{\alpha_{n+1}} \,\mathrm{d}\boldsymbol{x} \\ &= \frac{|T|}{|\widehat{T}|} \int_{\widehat{T}} \widehat{\lambda}_1^{\alpha_1} \widehat{\lambda}_2^{\alpha_2} \cdots \widehat{\lambda}_{n+1}^{\alpha_{n+1}} \,\mathrm{d}\widehat{\boldsymbol{x}} \\ &= \frac{|T|}{|\widehat{T}|} \int_{\widehat{T}} \widehat{x}_1^{\alpha_1} \widehat{x}_2^{\alpha_2} \cdots \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^n \widehat{x}_i\right)^{\alpha_{n+1}} \,\mathrm{d}\widehat{\boldsymbol{x}}. \end{aligned}$$

Since

$$\begin{aligned} \widehat{T} &= \left\{ \widehat{\boldsymbol{x}} \in \mathbb{R}^n : \widehat{x}_i \in [0,1], i = 1, \dots, n, \sum_{i=1}^n \widehat{x}_i \le 1 \right\} \\ &= \Bigg\{ \widehat{\boldsymbol{x}} \in \mathbb{R}^n : \widehat{x}_1 \in [0,1], \widehat{x}_2 \in [0, 1 - \widehat{x}_1], \\ &\qquad \widehat{x}_3 \in [0, 1 - (\widehat{x}_1 + \widehat{x}_2)], \dots, \widehat{x}_n \in \left[0, 1 - \sum_{i=1}^{n-1} \widehat{x}_i\right] \Bigg\} \end{aligned}$$

the integral becomes

$$I = \frac{|T|}{|\widehat{T}|} \int_0^1 \widehat{x}_1^{\alpha_1} \cdots \int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \ldots \mathrm{d}\widehat{x}_1.$$

We consider a more generic integral

$$\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi,$$

for $\alpha, \beta \in \mathbb{N}_0$ and $c \in [0,1]$. If $\beta = 0$ then

$$\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi = \frac{c^{\alpha+1}}{\alpha+1};$$

and when $\beta > 0$

$$\begin{aligned}
\int_0^c \xi^\alpha (c-\xi)^\beta \, \mathrm{d}\xi = \int_0^c \left(\frac{\xi^{\alpha+1}}{\alpha+1}\right)' (c-\xi)^\beta \, \mathrm{d}\xi &= \int_0^c \frac{\xi^{\alpha+1}}{\alpha+1} \beta (c-\xi)^{\beta-1} \, \mathrm{d}\xi = \ldots \\
&= \int_0^c \frac{\xi^{\alpha+\beta}}{(\alpha+1)\ldots(\alpha+\beta)} \beta! \, \mathrm{d}\xi \\
&= \frac{\alpha!\beta!}{(\alpha+\beta+1)!} c^{\alpha+\beta+1}.
\end{aligned}$$

Thus, we have be selecting $\alpha = \alpha_n$, $\beta = \alpha_{n+1}$, $\xi = \widehat{x}_n$ and $c = 1 - \sum_{i=1}^{n-1} \widehat{x}_i$ that

$$\int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \left(1 - \sum_{i=1}^{n-1} \widehat{x}_i\right)^{\alpha_n + \alpha_{n+1} + 1}.$$

Similarly, we have that

$$\begin{aligned}
&\int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \int_0^{1-\sum\limits_{i=1}^{n-1} \widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \\
&\qquad = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \int_0^{1-\sum\limits_{i=1}^{n-2} \widehat{x}_i} \widehat{x}_{n-1}^{\alpha_{n-1}} \left(1 - \sum_{i=1}^{n-1} \widehat{x}_i\right)^{\alpha_n + \alpha_{n+1} + 1} \mathrm{d}\widehat{x}_{n-1} \\
&\qquad = \frac{\alpha_n! \alpha_{n+1}!}{(\alpha_n + \alpha_{n+1} + 1)!} \frac{\alpha_{n-1}! (\alpha_n + \alpha_{n+1} + 1)!}{(\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2)!} \left(1 - \sum_{i=1}^{n-2} \widehat{x}_i\right)^{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2} \\
&\qquad = \frac{\alpha_{n-1}! \alpha_n! \alpha_{n+1}!}{(\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2)!} \left(1 - \sum_{i=1}^{n-2} \widehat{x}_i\right)^{\alpha_{n-1} + \alpha_n + \alpha_{n+1} + 2}.
\end{aligned}$$

Recursively, we get that for $k = 1, \dots, n$,

$$\int_0^{1-\sum_{i=1}^{n-k}\widehat{x}_i} \widehat{x}_{n-k+1}^{\alpha_1} \cdots \int_0^{1-\sum_{i=1}^{n-1}\widehat{x}_i} \widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{x}_n \, \mathrm{d}\widehat{x}_{n-1} \dots \mathrm{d}\widehat{x}_{n-k+1}$$
$$= \frac{\alpha_{n-k+1}! \dots \alpha_{n+1}!}{(\alpha_{n-k+1} + \cdots + \alpha_{n+1} + k)!} \left(1 - \sum_{i=1}^{n-k} \widehat{x}_i\right)^{\alpha_{n-k+1}+\cdots+\alpha_{n+1}+k}.$$

Therefore, we have that

$$\int_{\widehat{T}} \widehat{x}_1^{\alpha_1}\widehat{x}_2^{\alpha_2}\cdots\widehat{x}_n^{\alpha_n} \left(1 - \sum_{i=1}^{n} \widehat{x}_i\right)^{\alpha_{n+1}} \mathrm{d}\widehat{\boldsymbol{x}} = \frac{\alpha_1! \dots \alpha_{n+1}!}{(\alpha_1 + \cdots + \alpha_{n+1} + n)!}.$$

By setting $\alpha_1 = \cdots = \alpha_{n+1} = 0$ we have that

$$|\widehat{T}| = \int_{\widehat{T}} 1 \, \mathrm{d}\widehat{\boldsymbol{x}} = \frac{1}{n!};$$

hence,

$$I = \frac{\alpha_1! \dots \alpha_{n+1}! n!}{(\alpha_1 + \cdots + \alpha_{n+1} + n)!} |T|.$$

2. (2 points) Prove Theorem 12 from the lecture:

Let $\{\mathcal{T}_h\}$ be a family of triangulations of a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz-continuous boundary and $\{X_h\}$ the family of the corresponding finite element spaces under the assumptions required for convergence of a discrete solution. We assume all finite elements (T, P_T, Σ_T) are affine-equivalent to a reference element $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ with invertible affine mapping $F_T(\widehat{x}) = \mathbb{B}_T\widehat{x} + b_T$ such that $F_T(\widehat{T}) = T$.

Consider the quadrature formula

$$\int_{\widehat{T}} \widehat{\varphi} \, \mathrm{d}\boldsymbol{x} \approx \sum_{\ell=1}^{L} \widehat{\omega}_\ell \widehat{\varphi}(\widehat{b}_\ell)$$

where $\widehat{\varphi}$ and $\widehat{b}_\ell$, for $\ell = 1, \dots, L$, are the quadrature weights and nodes, defined on the reference element $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$, let $m \in \mathbb{N}$ be such that $\widehat{P} \subset P_m(\widehat{T})$, and let there hold at least one of the following conditions

(I) $\bigcup_{\ell=1}^{L} \{\widehat{b}_\ell\}$ contains a $P_{m-1}(\widehat{T})$-unisolvent set, or

(II) $\widehat{E}(\widehat{\varphi}) = 0$ for all $\widehat{\varphi} \in P_{2m-2}(\widehat{T})$, where

$$\widehat{E}(\widehat{\varphi}) = \int_{\widehat{T}} \widehat{\varphi}\,\mathrm{d}\boldsymbol{x} - \sum_{\ell=1}^{L} \widehat{\omega}_\ell \widehat{\varphi}(\widehat{b}_\ell).$$

Define $V_h = \{v_h \in X_h : \phi(v_h) = 0 \quad \forall \phi \in \Sigma_h^{\partial\Omega}\} \subset H_0^1(\Omega) \cap C(\overline{\Omega})$ and the bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ as

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right) (b_{\ell_T}),$$

where $\omega_{\ell,T} = |\det \mathbb{B}_T| \widehat{\omega}_\ell$, $b_{\ell,T} = F_T(\widehat{b}_\ell)$, $a_{ij} \in L^\infty(\Omega)$ and let there exist a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

Prove that the bilinear form a_h is *uniformly V_h-elliptic*; i.e., there exists a constant $\widetilde{\alpha} > 0$ independent of h such that

$$a_h(v_h, v_h) \geq \widetilde{\alpha}|v_h|_{1,h}^2 \qquad \forall v_h \in V_h, \forall h > 0.$$

Hint. First show that there exists a constant $\widehat{C} > 0$ such that

$$\widehat{C}|\widehat{p}|_{1,\widehat{T}}^2 \leq \sum_{\ell=1}^{L} \widehat{\omega}_\ell |\widehat{\nabla}\widehat{p}(\widehat{b}_\ell)|^2 \qquad \forall \widehat{p} \in \widehat{P} \tag{2.2}$$

for condition (I) and (II) separately.

Solution:

Assume (I) holds, then, for any $p \in \widehat{P}$

$$\begin{aligned}
&\sum_{\ell=1}^{L} \widehat{\omega}_\ell |\widehat{\nabla}\widehat{p}(\widehat{b}_\ell)|^2 = 0 \\
\Longrightarrow \quad & \left(\widehat{\nabla}\widehat{p}\right)(\widehat{b}_\ell) = 0 \quad \ell = 1, \ldots, L, \qquad (\text{as } \widehat{\omega}_\ell > 0) \\
\Longrightarrow \quad & \widehat{\nabla}\widehat{p} = 0 \qquad \left(\text{since } \widehat{\nabla}\widehat{p} \in \left[P_{m-1}(\widehat{T})\right]^n\right);
\end{aligned}$$

therefore, $\widehat{p} \in \widehat{P}_0(\widehat{T})$. Hence, the mapping

$$\widehat{p} \to \left[\sum_{\ell=1}^{L} \widehat{\omega}_\ell |\widehat{\nabla}\widehat{p}(\widehat{b}_\ell)|^2\right]^{1/2}$$

is a norm on the factor space $\widehat{P}/P_0(\widehat{T})$. Since $|\cdot|_{1,\widehat{T}}$ is also a norm on this factor space (2.2) follows by equivalence of norms on finite dimensional spaces.

If, instead, (II) holds then

$$0 = \widehat{E}(|\widehat{\nabla}\widehat{p}|^2) = \int_{\widehat{T}} |\widehat{\nabla}\widehat{p}|^2 \, \mathrm{d}\widehat{\boldsymbol{x}} - \sum_{\ell=1}^{L} \widehat{\omega}_\ell |\widehat{\nabla}\widehat{p}(\widehat{b}_\ell)|^2.$$

Hence, (2.2) holds as an equality for $\widehat{C} = 1$.

Now we have proven the hint we consider any $v_h \in V_h$ and $T \in \mathcal{T}_h$, define $F_T(\widehat{\boldsymbol{x}}) = \mathbb{B}_T\widehat{\boldsymbol{x}} + \boldsymbol{b}_T$ as the invertible affine map of $\widehat{T}$ to T, and denote $p_T = v_h|_T$ and $\widehat{p}_T = p_T \circ F_T$. Then, $\widehat{p}_T \in \widehat{P}$ and

$$\frac{\partial \widehat{p}_T}{\partial \widehat{x}_i}(\widehat{\boldsymbol{x}}) = \sum_{j=1}^{n} \frac{\partial p_T}{\partial x_j}(F_T(\widehat{\boldsymbol{x}})(B_T)_{ji}) \qquad \Longrightarrow \qquad |\widehat{\nabla}\widehat{p}_T(\widehat{\boldsymbol{x}})| \le |\nabla p_T(F_T(\widehat{\boldsymbol{x}}))| \|\mathbb{B}_T\|. \tag{2.3}$$

Similarly,

$$|\nabla p_T(\boldsymbol{x})| \le |\widehat{\nabla}\widehat{p}_T(F_T^{-1}(\boldsymbol{x}))| \|\mathbb{B}_T^{-1}\|.$$

Hence,

$$\begin{aligned} |p_T|_{1,T}^2 = \int_T |\nabla p_T(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} &= |\det \mathbb{B}_T| \int_{\widehat{T}} |(\nabla p_T)(F_T(\widehat{\boldsymbol{x}}))|^2 \, \mathrm{d}\widehat{\boldsymbol{x}} \\ &\le |\det \mathbb{B}_T| \|\mathbb{B}_T^{-1}\|^2 \int_{\widehat{T}} |(\widehat{\nabla}\widehat{p}_T)(\widehat{\boldsymbol{x}})|^2 \, \mathrm{d}\widehat{\boldsymbol{x}} \\ &= |\det \mathbb{B}_T| \|\mathbb{B}_T^{-1}\|^2 |\widehat{p}_T|_{1,\widehat{T}}^2. \end{aligned} \tag{2.4}$$

Therefore,

$$\begin{aligned} \sum_{\ell=1} \omega_{\ell,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial v_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right)(b_{\ell,T}) &= \sum_{\ell=1} \omega_{\ell,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial p_T}{\partial x_i} \frac{\partial p_T}{\partial x_j} \right)(b_{\ell,T}) \\ &\ge \theta \sum_{\ell=1} \omega_{\ell,T} |\nabla p_T(b_{\ell,T})|^2 && \text{by (2.1)} \\ &\ge \theta |\det \mathbb{B}_T| \|\mathbb{B}_T\|^{-2} \sum_{\ell=1} \widehat{\omega}_{\ell,T} |\widehat{\nabla}\widehat{p}_T(\widehat{b}_\ell)|^2 && \text{by (2.3)} \\ &\ge \widehat{C}\theta |\det \mathbb{B}_T| \|\mathbb{B}_T\|^{-2} |\widehat{p}_T|_{1,\widehat{T}}^2 && \text{by (2.2)} \\ &\ge \widehat{C}\theta \left(\|\mathbb{B}_T\| \|\mathbb{B}_T^{-1}\| \right)^{-2} |p_T|_{1,T}^2 && \text{by (2.4).} \end{aligned}$$

As

$$\|\mathbb{B}_T\| \|\mathbb{B}_T^{-1}\| \le \frac{h_T}{\widehat{\rho}} \frac{\widehat{h}}{\rho_T} \le \frac{\widehat{h}\sigma}{\widehat{\rho}}$$

then,

$$\sum_{\ell=1} \omega_{\ell,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial v_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} \right)(b_{\ell,T}) \ge \widehat{C}\theta \left(\frac{\widehat{\rho}}{\widehat{h}\sigma} \right)^2 |p_T|_{1,T}^2$$

Summation over all $T \in \mathcal{T}_h$ and setting $\widetilde{\alpha} = \widehat{C}\theta\widehat{\rho}^2/(\widehat{h}\sigma)^2$ completes the proof.

3. (2 points) Let $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$ be a finite element, and let $l, m \in \mathbb{N}_0$ and $r, q \in [1, \infty]$ be such that $l \le m$ and $\widehat{P} \subset W^{l,r}(\widehat{T}) \cap W^{m,q}(\widehat{T})$. For any $T \in \mathcal{T}_h$, let (T, P_T, Σ_T) be a finite element which is affine-equivalent to $(\widehat{T}, \widehat{P}, \widehat{\Sigma})$. Then, there exists a positive constant C, depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, and n such that

$$|v|_{m,q,T} \le C \frac{h_T^l}{\varrho_T^m} |T|^{\frac{1}{q}-\frac{1}{r}} |v|_{l,r,T}, \qquad \text{for all } v \in P_T, T \in \mathcal{T}_h. \tag{3.5}$$

Let X_h be the finite element space corresponding to $\mathcal{T}_h$ and the finite elements (T, P_T, Σ_T). Introduce the seminorms

$$|v|_{m,q,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,q,T}^q \right)^{1/q} \quad \text{if } q < \infty, \qquad |v|_{m,\infty,h} = \max_{T \in \mathcal{T}_h} |v|_{m,\infty,T}.$$

Let $\mathcal{T}_h$ satisfy

$$\frac{h_T}{\varrho_T} \le \sigma, \qquad \text{for all } T \in \mathcal{T}_h,$$

and the *inverse assumption*

$$\exists \kappa > 0 : \qquad \frac{h}{h_T} \le \kappa \quad \text{for all } T \in \mathcal{T}_h,$$

where $h = \max_{T \in \mathcal{T}_h} h_T$. Prove that the *inverse inequality*

$$|v_h|_{m,q,h} \le C h^{l-m+\min(0, n/q - n/r)} |v_h|_{l,r,h}, \qquad \text{for all } v_h \in X_h,$$

where C is a positive constant depending only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, κ, and Ω.

Hint. The following inequalities may be useful.

Hölder Inequality For any non-negative numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$

$$\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n a_i^p \right)^{1/p} \left(\sum_{i=1}^n b_i^q \right)^{1/q}$$

for any $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$.

Jensen Inequality For any non-negative numbers $a_1, \dots, a_n$

$$\left(\sum_{i=1}^n a_i^q \right)^{1/q} \le \left(\sum_{i=1}^n a_i^p \right)^{1/p}$$

for any $p, q \in (0, \infty)$ satisfying $p \le q$.

Solution:

From (3.5), the fact that there exists constants σ and C_2 such that for all $T \in \mathcal{T}_h$

$$\frac{h_T}{\varrho_T} \leq \sigma \qquad \text{and} \qquad |T| \leq C_2 h_T^n \quad \text{(from Homework 3, q.2(a))},$$

and the *inverse assumption* we have that

$$|v_h|_{m,q,T} \leq \overline{C} h^{l-m} h^{n/q-n/r} |v_h|_{l,r,T}, \qquad \text{for all } v_h \in X_h, T \in \mathcal{T}_h,$$

where $\overline{C}$ depends only on $\widehat{T}$, $\widehat{P}$, l, m, r, q, n, σ, and κ. We also note that from Homework 3, q.2(a) there exists a constant C_1 such that $C_1 h_T^n \leq |T|$ and, hence,

$$\operatorname{card} \mathcal{T}_h \leq \frac{|\Omega|}{\min_{T\in\mathcal{T}_h} |T|} \leq \frac{|\Omega|}{C_1 \min_{T\in\mathcal{T}_h} h_T^n} \leq \frac{|\Omega|\kappa^n}{C_1} h^{-n} = \widetilde{C} h^{-n},$$

where $\widetilde{C}$ depends only on n, σ, κ, and Ω.

We now consider four separate cases:

- $q = \infty$: There exists a $T_0 \in \mathcal{T}_h$ such that $|v_h|_{m,\infty,h} = |v_h|_{m,\infty,T_0}$; therefore,

$$|v_h|_{m,\infty,h} \leq \overline{C} h^{l-m} h^{-n/r} |v_h|_{l,r,T_0} \leq \overline{C} h^{l-m-n/r} |v_h|_{l,r,h}$$

- $r = \infty$:

$$\begin{aligned} |v_h|_{m,q,h} &\leq \overline{C} h^{l-m} h^{n/q} \left(\sum_{T\in\mathcal{T}_h} |v|_{l,\infty,T}^q \right)^{1/q} \\ &\leq \overline{C} h^{l-m} h^{n/q} \left(\operatorname{card} \mathcal{T}_h\right)^{1/q} |v_h|_{l,\infty,h} \leq \overline{C}\widetilde{C}^{1/q} h^{l-m} |v_h|_{l,\infty,h} \end{aligned}$$

- $r \leq q < \infty$: By Jensen inequality

$$|v_h|_{m,q,h} \leq \overline{C} h^{l-m+n/q-n/r} \left(\sum_{T\in\mathcal{T}_h} |v|_{l,r,T}^q \right)^{1/q} \leq \overline{C} h^{l-m+n/q-n/r} |v_h|_{l,r,h}$$

- $q < r < \infty$: By Hölder inequality

$$\begin{aligned} |v_h|_{m,q,h} &\leq \overline{C} h^{l-m+n/q-n/r} \left(\sum_{T\in\mathcal{T}_h} 1 \cdot |v|_{l,r,T}^q \right)^{1/q} \\ &\leq \overline{C} h^{l-m+n/q-n/r} \left(\left(\sum_{T\in\mathcal{T}_h} 1 \right)^{1-q/r} \left(\sum_{T\in\mathcal{T}_h} |v|_{l,r,T}^{q\cdot r/q} \right)^{q/r} \right)^{1/q} \\ &\leq \overline{C} h^{l-m+n/q-n/r} \left(\operatorname{card} \mathcal{T}_h\right)^{1/q-1/r} |v_h|_{l,r,h} \\ &\leq \overline{C}\widetilde{C}^{1/q-1/r} h^{l-m} |v_h|_{l,r,h} \end{aligned}$$