Variational Formulation

(1) $-\Delta u = f$ in $\Omega \subset \mathbb{R}^n$ (bounded domain)
$u = 0$ on $\partial\Omega$ (homogeneous Dirichlet)

Ω bounded domain with Lipschitz continuous boundary $\partial\Omega$.

· Classical solution: $u \in C^2(\Omega) \cap C(\overline{\Omega})$
$\Rightarrow f \subset C(\Omega)$

- required regularity - very strong, often not satisfied
Hence, try to formulate in a weaker sense
↳ variational formulation

Multiple by test function $v \in C_0^\infty(\Omega)$

$$-\int_\Omega \Delta u \, v \, dx = \int_\Omega f v \, dx$$

Apply Green's identity

$$\int_\Omega \Delta u v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v dx \quad \text{as } v = 0 \text{ on } \partial\Omega$$

Any classical solution satisfies above equation for any $v \in C_0^\infty(\Omega)$

Don't require u to have second order derivatives, and derivatives can be defined only pointwise.

Integrals have sense for $u, v \in H^1(\Omega)$, $f \in L^2(\Omega)$.

A weak solution of ① is a function $u \in H_0^1(\Omega)$ such that

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx}_{a(u,v)} = \underbrace{\int_\Omega f v \, dx}_{\langle F, v \rangle} \quad \forall v \in \underbrace{H_0^1(\Omega)}_{V}$$

Can show that:

- a is continuous on V:

$$a(u,v) \leq \|\nabla u\|_{0,\Omega} \|\nabla v\|_{0,\Omega} \quad \text{(Hölder inequality)}$$
$$= |u|_{1,\Omega} |v|_{1,\Omega} \quad (|\cdot|_{1,\Omega} \text{ norm in } H_0^1(\Omega))$$

- a is V-elliptic on V:

$$a(v,v) = \int_\Omega |\nabla v|^2 dx = |v|_{1,\Omega}^2$$

- $F \in V'$:

$$|\langle F, v \rangle| \leq \|f\|_{0,\Omega} \|v\|_{0,\Omega} \leq C \|f\|_{0,\Omega} |v|_{1,\Omega} \quad \forall v \in V$$

Properties often valid for other 2nd order elliptic PDEs.

The weak solution of ① exists and is unique.
Any classical solution of ① is a weak solution, and if a weak solution satisfies $u \in C^2(\Omega) \cap C(\bar{\Omega})$ then it is a classical solution.

Abstract Variational Problem (AVP)

Find $u \in V$ such that

$$a(u,v) = \langle f, v \rangle \quad \forall v \in V$$

where V is a real Hilbert space with norm $\|\cdot\|_V$, f is a continuous linear functional on V, and $a: V \times V \to \mathbb{R}$ is a bilinear form which is

a) continuous; i.e, $\exists$ positive constant $M>0$ such that

$$|a(u,v)| \leq M \|u\|_V \|v\|_V \quad \forall u,v \in V$$

b) V-elliptic; i.e, $\exists$ positive constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V.$$

Lax-Milgram Lemma The abstract variational problem has a unique solution.

Proof (Multiple proofs, use Banach F.P.):

Let $(\cdot,\cdot)$ be an inner product in V. By Riesz representation theorem

$$\exists!\ F \in V : \langle f, v \rangle = (F, v) \quad \forall v \in V$$

Similarly, $\forall u \in V\ \exists!\ Au \in V : a(u,v) = (Au, v)$ $\forall v \in V$ since $a(u,\cdot) \in V'$.

$A: V \to V$ is linear and

$$\|Au\|_V^2 = a(u, Au) \leq M \|u\|_V \|Au\|_V$$

$$\Rightarrow \|Au\|_V \leq M \|u\|_V \quad \forall u \in V$$

$$\Rightarrow A \text{ is continuous and } \|A\|_{\mathcal{L}(V,V)} \leq M$$

The solution of AVP is equivalent to the solution of $Au = F$. Let $\rho > 0$ and define

$$T_\rho v = v - \rho(Av - F).$$

Then,

$$Au = F \iff T_\rho u = u \quad \text{i.e } u \text{ is a F.P of } T_\rho.$$

Use Banach F.P, so T_ρ must be contractive:

$$\|v - \rho A v\|_V^2 = (v - \rho Av, v - \rho Av)$$

$$= \|v\|^2 - 2\rho \underbrace{(v, Av)}_{a(v,v) \geq \alpha \|v\|_V^2} + \rho^2 \|Av\|^2$$

$$\leq \|v\|^2 (1 - 2\rho + \rho^2 M^2)$$

$$\Rightarrow 1 - 2\alpha\rho + \rho^2 M^2 < 1 \iff \rho \in \left(0, \frac{2\alpha}{M^2}\right)$$

$\Rightarrow T_\rho$ is contractive and has unique fixed point □

<u>Remark</u> The operator A maps V onto V and

$$\alpha \|u\|_V^2 \leq a(u,u) = (Au, u) \leq \|Au\|_V \|u\|_V$$

$$\Rightarrow \|u\|_V \leq \frac{1}{\alpha} \|Au\|_V = \frac{1}{\alpha} \|f\|_V$$

(solution continuously depends on the data)

$$\Rightarrow A^{-1} \in \mathcal{L}(V,V) \quad \& \quad \|A^{-1}\|_{\mathcal{L}(V,V)} \leq \frac{1}{\alpha}$$

If u_1, u_2 are solutions corresponding to $f_1, f_2 \in V$ then $\|u_1 - u_2\| \leq \frac{1}{\alpha} \|f_1 - f_2\|$

<u>The AVP is well posed</u> = solution exists, is unique, and continuously depends on the data.

Theorem Let the bilinear form a from the AVP be symmetric. Then, the solution of the AVP is equivalent to the minimisation of the quadratic functional

$$J(v) = \frac{1}{2} a(v,v) - \langle f, v \rangle$$

Proof Let u be the solution of the AVP. Then, $\forall v \in V$

$$J(u+v) = \frac{1}{2} a(u,u) + \frac{1}{2} a(u,v) + \frac{1}{2}(v,u) + \frac{1}{2} a(v,v) - \langle f,u \rangle - \langle f,v \rangle$$

$$= J(u) + 0 + \frac{1}{2} a(v,v)$$

$$\geq J(u) + \frac{\alpha}{2} \|v\|_V^2 > J(u) \quad \text{if } v \neq 0$$

$\Rightarrow$ $J(v)$ obtains minimum exactly at u

$\Rightarrow$ u unique solution of minimisation problem

$$J(u) = \inf_{v \in V} J(v)$$

$\square$

Galerkin method

Approximate the space V by finite dimensional subspaces V_h and for each V_h define an approximate solution $u_h \in V_h$ (discrete solution) as solution of the discrete problem: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h$$

Lax-Milgram Lemma $\Rightarrow$ $\exists! \, u_h \in V_h$

If $a(\cdot,\cdot)$ is symmetric, then can define the approximate solution as the solution of the minimization problem $J(u_h) = \inf_{v_h \in V_h} J(v_h)$

(Ritz method)

Let $\{\varphi_1, \dots, \varphi_N\}$ be a basis in V_h. Then,

$$u_h = \sum_{j=1}^{N} u_j \varphi_j$$

$$\Rightarrow \sum_{j=1}^{N} a(\varphi_j, \varphi_i) u_j = \langle f, \varphi_i \rangle \qquad i = 1, \dots, N$$

(equivalent to previous - if holds for all basis $\varphi_i \in V_h \Rightarrow \forall v \in V_h$).

Gives system of linear equations for unknowns u_j.

Denote $U = (u_1, \dots, u_N)^T$, $F = (\langle f, \varphi_1 \rangle, \dots \langle f, \varphi_N \rangle)^T$ and $A = (a_{ij})_{i,j=1}^{N}$, $a_{ij} = a(\varphi_j, \varphi_i)$. Then,

u_h is a discrete solution $\Leftrightarrow U$ solution of $AU = F$.

(unique solution due to equivalence of discrete problem)

Can show that

- A is positive definite:

$$U \cdot AU = \sum_{i,j=1}^{N} u_i a_{ij} u_j = a(u_j \varphi_j, u_i \varphi_i)$$

$$= a\left(\sum_{j=1}^{N} u_j \varphi_j, \sum_{i=1}^{N} u_i \varphi_i\right)$$

$$\geq \alpha \left\| \sum_{i=1}^{N} u_i \varphi_i \right\|_V^2$$

$$> 0 \quad \text{as } \varphi_i \text{ linearly independent if } U \neq 0.$$

- if a is symmetric then A is symmetric
  $\Rightarrow A$ is SPD
  $\Rightarrow$ linear system can be solved by using conjugate gradient method

A is usually called the stiffness matrix (comes from application in elasticity).

For the following we assume AVP corresponds to a valid weak formulation of a 2^{nd} order (or 4^{th} order) elliptic PDE defined in a bounded domain with Lipschitz continuous boundary
$\Rightarrow V$ will be a subspace of $H^1(\Omega)$ or $H^2(\Omega)$.

Finite element method is its simplest form is just the Galerkin method with special construction of the space V_h which are called the finite element spaces.