We can define a finite element by a triple:

(definition for n-simplex in blue)

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- $T \subset \mathbb{R}^n$ – Set in $\mathbb{R}^n$ defining element shape

  T = n-simplex with vertices $a_1, \dots, a_{n+1}$

- P_T – Finite dimensional space of functions on T

  $P_T = P_k(T)$

- Σ_T – Degrees of freedom. Set of functionals uniquely determining P_T

  $\Sigma_T = \{p(z) : z \in L_k(T)\}$ where L_k = principal lattice

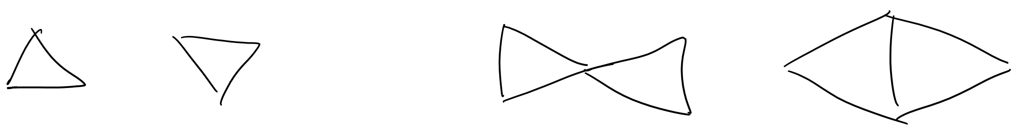
  (any function in P_T uniquely determined by values at principal lattice vertices).

Let $\Omega \subset \mathbb{R}^n$ bounded domain with Lipschitz continuous boundary which we assume is polyhedral (can be decomposed into n-simplices).

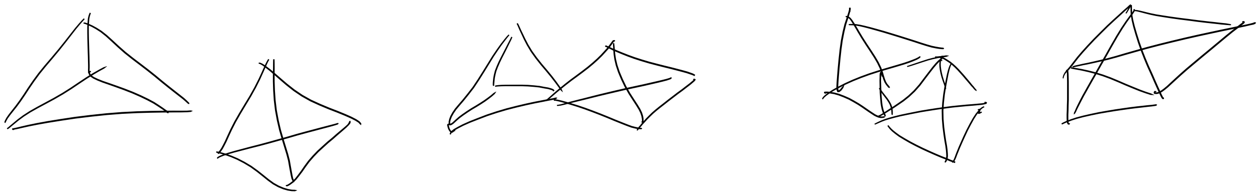
Let $\mathcal{T}_h$ be a triangulation of Ω satisfying $(\mathcal{T}_h 1)$-$(\mathcal{T}_h 4)$ consisting of n-simplices. In addition we assume

$(\mathcal{T}_h 5)$ Any two different n-simplices $T, \tilde{T} \in \mathcal{T}_h$ are either disjoint or their intersection is a common m-face of both n-simplices

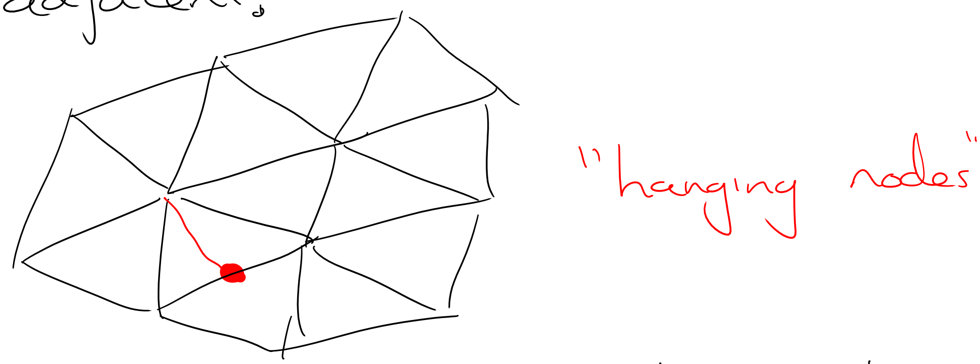
<u>Ex.</u> $n=2$ $\quad T \cap \tilde{T} = \emptyset$ or 'common vertex' or 'common edge'



$n=3$ $\quad T \cap \tilde{T} = \emptyset$ or 'common vertex' or 'common edge' or 'common face'



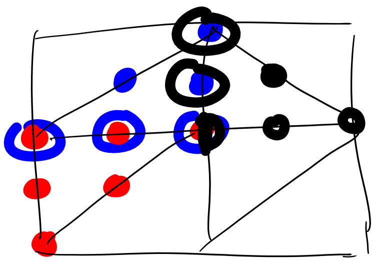
Equivalent form of $(\mathcal{T}_h 5)$: any face of any element of $\mathcal{T}_h$ is either a subset of $\partial\Omega$ or a face of another element of $\mathcal{T}_h$ $\rightarrow$ call these elements 'adjacent'.



Assume that we are given a triangulation $\mathcal{T}_h$ satisfying $(\mathcal{T}_h 1)-(\mathcal{T}_h 5)$ consisting of n-simplices and that the finite element (T, P_T, Σ_T) defined in (I) is assigned to each $T \in \mathcal{T}_h$ (with same k).

Need to construct finite element space to glue P_T from each element together.

Let us denote by $M_h = \bigcup_{T \in \mathcal{T}_h} L_k(T)$



If F is any m-face of an element $T \in \mathcal{T}_h$, $m \in \{0, \dots, n-1\}$ then $L_k(T) \cap F = L_k(F)$

Let $\tilde{T} \in \mathcal{T}_h$, $T \cap \tilde{T} \neq \emptyset$, then $L_k(\tilde{T}) \cap F \subset \tilde{T} \cap T = \tilde{F}$

$$\Rightarrow L_k(\tilde{T}) \cap F = (L_k(\tilde{T}) \cap \tilde{F}) \cap F$$
$$= L_k(\tilde{F}) \cap F \subset L_k(T) \cap F = L_k(F)$$

Consequently

$$M_h \cap F = L_k(F) = L_k(T) \cap F \qquad (+)$$

Similarly, $L_k(\tilde{T}) \cap T = L_k(\tilde{T}) \cap \tilde{F} = L_k(\tilde{F}) \subset L_k(T)$

Thus, $M_h \cap T = L_k(T)$ $\qquad (*)$

Let $N_h = \text{card}\, M_h$ and $M_h = \{z_i\}_{i=1}^{N_h}$. Denote by $\mathcal{T}_h^i = \{T \in \mathcal{T}_h : z_i \in T\}$. Let $\alpha_1, \dots, \alpha_{N_h} \in \mathbb{R}$ be any real numbers. Due to $(*)$ and previous theorem on $L_k(T)$ there exists a uniquely determined function $v_h \in L^2(\Omega)$ such that for any $T \in \mathcal{T}_h$

$$v_h|_T \in P_T \quad \text{and} \quad v_h|_T(z_i) = \alpha_i \quad \underbrace{\forall z_i \in T, i = 1, \dots, N_h}_{L_k(T) \text{ due to } (*)}$$

If $T, \tilde{T} \in \mathcal{T}_h^i$, then $(v_h|_T)(z_i) = \alpha_i = (v_h|_{\tilde{T}})(z_i)$.

If we consider functions v_h for all possible values of α_i we obtain the space

$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in P_T \ \forall T \in \mathcal{T}_h,$$
$$v_h|_T(z_i) = v_h|_{\tilde{T}}(z_i) \quad \forall T, \tilde{T} \in \mathcal{T}_h^i, i = 1, \dots, N_h \}$$

This is the finite element space assigned to $\mathcal{T}_h$ and (T, P_T, Σ_T). The set of degrees of freedom of X_h is $\Sigma_h = \{v_h(z) : z \in M_h\}$.

Possible to show from continuity at lattice points that functions from X_h are continuous over Ω.

Theorem The space X_h defined above satisfies

$$X_h \subset C(\bar{\Omega}) \underbrace{\cap H^1(\Omega)}_{\text{from previous theorem}}$$

Proof Consider any $v_h \in X_h$ and $x \in \bar{\Omega}$. Then $\exists T \in \mathcal{T}_h : x \in T$. If $x \in \operatorname{int} T$ or $x \in \operatorname{int}(\partial T \cap \partial \Omega)$ then v_h is continuous at x since $v_h|_T \in P_T \subset C(\bar{T})$.

If $x \in \partial T \setminus \operatorname{int}(\partial T \cap \partial \Omega)$; then, $\exists \tilde{T} \in \mathcal{T}_h$, $\tilde{T} \neq T$, such that $x \in T \cap \tilde{T} =: F$ (common m-face, $m \in \{0, \dots, n-1\}$).

$\Rightarrow (v_h|_T)|_F$ is uniquely determined by its values at points from $L_k($

$(v_h|_{\tilde{T}})|_F$

According to $(+)$ and definition of X_h, the function $v_h|_T$ and $v_h|_{\tilde{T}}$ have the same values on $L_k(T)$

$$\Rightarrow (v_h|_T)|_F = (v_h|_{\tilde{T}})|_F$$

$$\Rightarrow (v_h|_T)(x) = (v_h|_{\tilde{T}})(x)$$

(T was arbitrary, so valid for all $T \in \mathcal{T}_h$)

$\Rightarrow v_h$ is continuous.

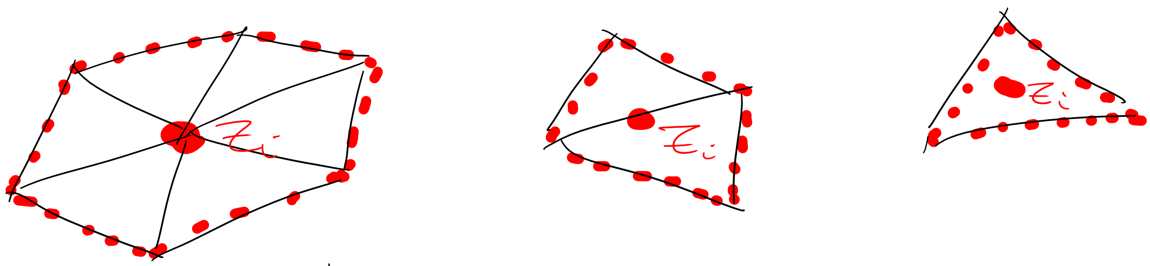
Define functions $\varphi_1, \dots, \varphi_{N_h} \in X_h$ by

$$\varphi_i(z_j) = \delta_{ij} \qquad i,j = 1, \dots, N_h$$

$\Rightarrow \{\varphi_i\}_{i=1}^{N_h}$ is a basis of X_h

and $\operatorname{supp} \varphi_i \subset \bigcup_{T \in \mathcal{T}_h^i} T \quad \Rightarrow$ basis functions have "small" support.

Ex: $\operatorname{supp} \varphi_i$:



From Poisson problem

$$\int_\Omega \nabla \varphi_j \cdot \nabla \varphi_i \, dx = \int_{\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j} \nabla \varphi_j \cdot \nabla_i \, dx$$

and results in sparse matrix as $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$ in many cases.

Finite elements defined by n-rectangles

$$Q_k = \left\{ \sum_{\substack{\alpha_i \le k \\ i=1,\dots,n}} \gamma_\alpha x^\alpha, \gamma_\alpha \in \mathbb{R} \right\}$$ – space of polynomials of degree k with respect to each variable $x_1, \dots, x_n$

(Tensor product of 1D polynomials of degree k).

$\Rightarrow \dim Q_k = (k+1)^n$

$P_k \subset Q_k \subset P_{k \cdot n}$

$Q_k(A) = \{ p|_A : p \in Q_k \}, \ A \subset \mathbb{R}^n$

Theorem Every polynomial $p \in Q_k$ is uniquely determined by its values on the set

$$M_k = \left\{ x = \left(\frac{i_1}{k}, \frac{i_2}{k}, \dots, \frac{i_n}{k}\right) \in \mathbb{R}^n, i_1, \dots, i_n \in \{0, 1, \dots, k\} \right\}$$

Proof Since $\dim Q_k = \operatorname{card} M_k$ it is sufficient to show that

$\forall x \in M_k\ \exists p \in Q_k : p(z)=1$ and $p(y)=0\ \forall y \in M_k \setminus \{z\}$

(so these functions form a linear independent basis of Q_k)

This is easy to construct: let $z=\left(\frac{i_1}{k},\frac{i_2}{k},\dots,\frac{i_n}{k}\right)$ then

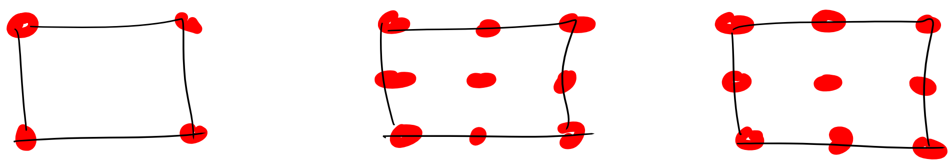
$$p(x)=\prod_{j=1}^{n}\ \prod_{\substack{i_j'=0\\ i_j'\neq i_j}}^{k} \frac{kx-i_j'}{i_j-i_j'}$$

(tensor product of 1D Lagrange polynomials) $\square$

Note: $\operatorname{conv} M_k=[0,1]^n$

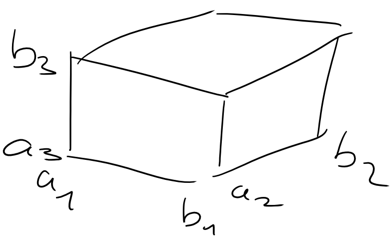
↳ Principal lattice of the unit hyper-cube

Ex: $k=1$ $\quad$ $k=2$ $\quad$ $k=3$



General n-rectangle:

$$T=\prod_{i=1}^{n}[a_i,b_i]=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_i\in[a_i,b_i],\ i=1,\dots,n\}$$

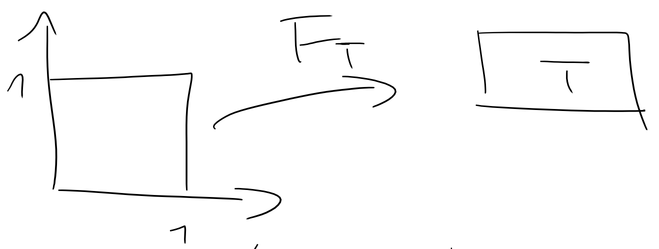


Faces: $\{a_j\}\times\prod_{\substack{i=1\\ i\neq j}}^{n}[a_i,b_i]$ & $\{b_j\}\times\prod_{\substack{i=1\\ i\neq j}}^{n}[a_i,b_i]$, $j=1,\dots,n$

Edges: $[a_j,b_j]\times\prod_{\substack{i=1\\ i\neq j}}^{n}\{c_i\}$, $j=1,\dots,n$ where $c_i\in\{a_i,b_i\}$

Vertices: $x=(x_1,\dots,x_n)$ where $x_i\in\{a_i,b_i\}$, $i=1,\dots,n$

For any n-rectangle T there is a mapping $F_T(x) = B_T x + b_T$, where $B_T \in \mathbb{R}^{n\times n}$ which is diagonal and $b_T \in \mathbb{R}^n$ such that $T = F_T([0,1]^n)$



Note that F_T is <u>not</u> unique (due to rotation)

$B_T = \mathrm{diag}(b_1 - a_1, b_2 - a_2, \dots, b_n - a_n)$, $b_T = (a_1, \dots, a_n)$

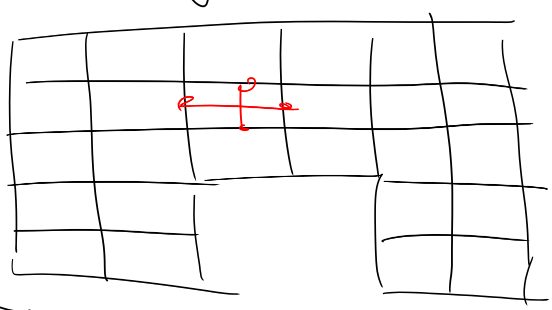
Any $p \in Q_k$ is uniquely determined also by its values on the set $M_k(T) = F_T(M_k)$

Clearly $M_k(T) \subset T$ (Principal lattice of order k for n-rectangle T).

We introduce finite elements called <u>n-rectangle of order k</u>:

$$T = \prod_{i=1}^{n} [a_i, b_i], \quad P_T = Q_k(T), \quad \Sigma_T = \{p(z) : z \in M_k(T)\}$$

Let Ω be an n-rectangle or a union of finitely many n-rectangles and let $\mathcal{T}_h$ be a triangulation of Ω consisting of n-rectangles.



Condition $(\mathcal{T}_h 5)$ can be formulated as: Each face of any n-rectangle $T \in \mathcal{T}_h$ is either a subset of $\partial\Omega$ or a face of another n-rectangle from $\mathcal{T}_h$

This condition implies that finite element spaces X_h defined analogously as for n-simplices satisfy $X_h \subset C(\overline{\Omega})$.

Examples of FEs considered so far define degrees of freedom as functions of some points

– such FEs are called <u>Lagrange</u> finite elements.