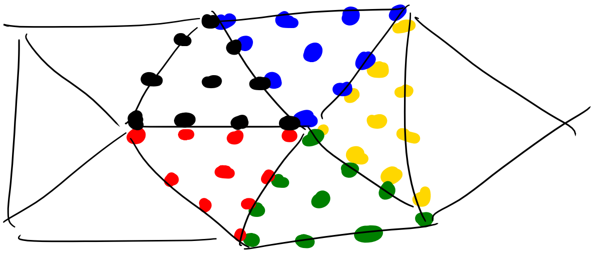
General definition of finite element space

Example Lagrange cubic triangle



Let $\Omega \subset \mathbb{R}^n$ be bounded domain with Lipschitz-continuous boundary and let $\mathcal{T}_h$ be a triangulation of Ω satisfying $(\mathcal{T}_h 1)-(\mathcal{T}_h 4)$. A finite element (T, P_T, Σ_T) is assigned to each $T \in \mathcal{T}_h$.

Let $T, \tilde{T} \in \mathcal{T}_h$ and $\phi \in \Sigma_T$, $\tilde{\phi} \in \Sigma_{\tilde{T}}$. We say that functionals ϕ and $\tilde{\phi}$ are equivalent if

$$\phi(v|_T) = \tilde{\phi}(v|_{\tilde{T}}) \qquad \forall v \in C^\infty(\mathbb{R}^n)$$

Assume for any $T \in \mathcal{T}_h$ no two functionals in Σ_T are equivalent.

The set $\bigcup_{T \in \mathcal{T}_h} \Sigma_T$ can be divided into subsets $\Sigma_{h,1}, \dots \Sigma_{h,N_h}$, where each subset contains only equivalent functionals, and any two equivalent functionals from $\bigcup_{T \in \mathcal{T}_h} \Sigma_T$ are in the same subset.

(Some sets may contain only one functional; for example, interior or boundary edge DoF in Lagrange FE).

We call the indices $i = 1, \dots, N_h$ (of $\Sigma_{h,i}$) <u>nodes</u>.

Let

$$\mathcal{T}_{h,i} = \{ T \in \mathcal{T}_h, \ \Sigma_{h,i} \cap \Sigma_T \neq \emptyset \}$$ (set of elements containing node i)

$$\Rightarrow \operatorname{card} \mathcal{T}_{h,i} = \operatorname{card} \Sigma_{h,i}$$

Denote by $\phi_{T,i}$ a functional in $\Sigma_{h,i} \cap \Sigma_T$

$$\Rightarrow \Sigma_{h,i} = \{ \phi_{T,i} : T \in \mathcal{T}_{h,i} \}$$

A finite element space can be defined by

$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in P_T \ \forall T \in \mathcal{T}_h, \ \phi_{T,i}(v_h|_T) = \phi_{\tilde{T},i}(v_h|_{\tilde{T}}) \ \forall T, \tilde{T} \in \mathcal{T}_{h,i}, \ i = 1, \dots, N_h \}$$

Denote

$$I_h(T) = \{ i \in \{1, \dots, N_h\}, \ T \in \mathcal{T}_{h,i} \}$$ (set of nodes contained in T).

$$\Rightarrow \operatorname{card} I_h(T) = \operatorname{card} \Sigma_T, \quad T \in \mathcal{T}_{h,i} \Leftrightarrow i \in I_h(T)$$

$$\Rightarrow \forall T \in \mathcal{T}_h : \Sigma_T = \{ \phi_{T,i} : i \in I_h(T) \}$$

Denote by $\{p_{T,i}\}_{i \in I_h(T)}$ the basis functions of P_T dual to the functionals $\{\phi_{T,i}\}_{i \in I_h(T)}$.

Set $p_{T,i} = 0$ in $\Omega \setminus T$, $i \in I_h(T)$.

Consider any $v_h \in X_h$. Then,

$$(*) \qquad v_h = \sum_{T \in \mathcal{T}_h} \sum_{i \in I_h(T)} \alpha_{T,i} \, p_{T,i}, \qquad \alpha_{T,i} \in \mathbb{R}$$

Since $v_h|_T = \sum_{i \in I_h(T)} \alpha_{T,i} \, p_{T,i}|_T \quad \forall T \in \mathcal{T}_h$ we have that

$$\alpha_{T,i} = \phi_{T,i}(v_h|_T)$$

From definition of X_h

$$(\star) \qquad \alpha_{T,i} = \alpha_{\tilde{T},i} \qquad \forall T, \tilde{T} \in \mathcal{T}_{h,i}, \ i = 1, \dots, N_h$$

Denote for $i=1,\dots,N_h$

$$\alpha_i = \alpha_{T,i} = \Phi_{T,i}(v_h|_T), \qquad T \in \mathcal{T}_{h,i}$$

Then,

$$v_h = \sum_{T\in\mathcal{T}_h} \sum_{i\in I_h(T)} \alpha_i p_{T,i} = \sum_{\substack{T\in\mathcal{T}_h,\, i\in\{1,\dots,N_h\}\\ T\in\mathcal{T}_{h,i}}} \alpha\, p_{T,i}$$

$$= \sum_{i=1}^{N_h} \alpha_i \boxed{\sum_{T\in\mathcal{T}_{h,i}} p_{T,i}} = p_i \in X_h$$

$\Rightarrow \{p_i\}_{i=1}^{N_h}$ are linearly independent and $\forall v_h \in X_h$

$$v_h = \sum_{i=1}^{N_h} \alpha_i p_i \quad \text{with} \quad \alpha_i = \Phi_{T,i}(v_h|_T),$$
$T\in\mathcal{T}_h$ arbitrary

$\Rightarrow \{p_i\}_{i=1}^{N_h}$ form basis of X_h & $\dim X_h = N_h$

(N_h = Number of sets of equivalent functionals)

We have that $\operatorname{supp} p_i \subset \bigcup_{T\in\mathcal{T}_{h,i}} T$

Define functionals $\{\Phi_i\}_{i=1}^{N_h}$ on X_h by

$(+)$ $\qquad \Phi_i(v_h) = \Phi_{T,i}(v_h|_T), \quad v_h \in X_h,$

where $T\in\mathcal{T}_{h,i}$ is arbitrary. Then,

$$\forall v_h \in X_h: \quad v_h = \sum_{i=1}^{N_h} \Phi_i(v_h)\, p_i.$$

The set $\Sigma_h := \{\Phi_i\}_{i=1}^{N_h}$ is the set of global degrees of freedom on X_h. Any $v_h\in X_h$ is uniquely determined by values $\{\Phi_i(v_h)\}_{i=1}^{N_h}$. Obviously $\Phi_i(p_j) = \delta_{ij}$ $\; i,j=1,\dots,N_h$

Alternatively:

$$\text{(X)} \qquad \phi_i(v_h) = \frac{1}{\operatorname{card} \mathcal{T}_{h,i}} \sum_{T \in \mathcal{T}_h} \phi_{T,i}(v_h|_T)$$

These functionals give same values on X_h as given by (+). Definition (X) advantageous for interpolation of functions which are, for example, not continuous across interfaces between elements of $\mathcal{T}_h$.

Let $Q(\Omega)$ be a space of real functions defined on $\bar{\Omega}$ such that $\forall T \in \mathcal{T}_h$ all functionals in Σ_T are defined on $Q(\Omega)|_T$. Furthermore, assume space $Q(\Omega)$ leads to same classes of equivalent functionals as space $C^\infty(\mathbb{R}^n)$; i.e., $\forall T, \tilde{T} \in \mathcal{T}_h$ and $\forall \phi \in \Sigma_T, \tilde{\phi} \in \Sigma_{\tilde{T}}$:

$$\phi(v|_T) = \tilde{\phi}(v|_{\tilde{T}}) \ \forall v \in C^\infty(\mathbb{R}^n) \Leftrightarrow \phi(v|_T) = \tilde{\phi}(v|_{\tilde{T}}) \ \forall v \in Q(\Omega)$$

This implies that $\phi_i(v) = \phi_{T,i}(v|_T) \quad \forall T \in \mathcal{T}_{h,i}, \ v \in Q(\Omega), i = 1, \ldots, N_h$

Define the mapping $\Pi_h : Q(\Omega) \to X_h$ by

$$\Pi_h v = \sum_{i=1}^{N_h} \phi_i(v) p_i, \quad v \in Q(\Omega)$$

If $X_h \subset Q(\Omega)$, then $\Pi_h v = v \quad \forall v \in X_h$; i.e., Π_h is a projection of $Q(\Omega)$ onto X_h.

$\Pi_h v$ is called the X_h-interpolant of v and Π_h is called the X_h-interpolation.

Theorem

$$(\Pi_h v)|_T = \Pi_T(v|_T) \quad \forall v \in Q(\Omega), T \in \mathcal{T}_h.$$

Proof

$$(\Pi_h v)|_T = \sum_{i=1}^{N_h} \phi_i(v) p_i|_T$$

$$= \sum_{i \in I_h(T)} \phi_i(v) p_i|_T$$

$$= \sum_{i \in I_h(T)} \phi_{T,i}(v|_T) p_{T,i}$$

$$= \Pi_T(v|_T)$$

Remark Important theorem tells us that global interpolation can be obtained by glueing together local interpolation together.

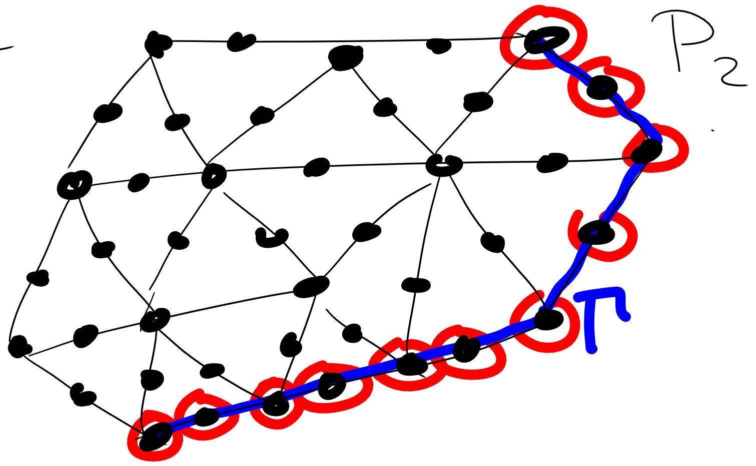
Boundary conditions

For function spaces from weak formulations of PDEs, Dirichlet BCs on some part $\Gamma \subset \partial\Omega$ lead to requirement that test functions should vanish on Γ; so for second-order PDEs we consider spaces of the type

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$$

Want to construct finite element spaces V_h which approximates V. Identify degrees of freedom corresponding to nods "on Γ": we define

$$\Sigma_h^\Gamma = \{\phi \in \Sigma_h : \phi(v|_\Omega) = 0 \ \forall v \in C^\infty(\mathbb{R}^n), v = 0 \text{ on } \Gamma\}$$

Ω 

Set $V_h = \{ v_h \in X_h : \phi(v_h) = 0 \ \forall \phi \in \Sigma_h^\Gamma \}$

If $Q(\Omega)$ is such that it makes sense to define

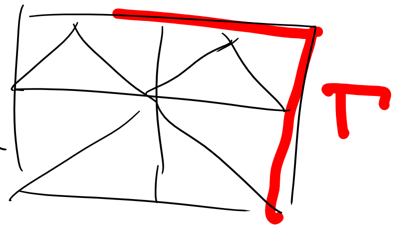
$$Q^\Gamma(\Omega) = \{ v \in Q(\Omega) : v|_\Gamma = 0 \}$$

and if $\phi(v) = 0 \quad \forall \phi \in \Sigma_h^\Gamma, v \in Q^\Gamma$

then Π_h maps $Q^\Gamma(\Omega)$ into V_h.

↳ will be assumed.

If $\mathcal{T}_h$ consists of polytopes and if any face $F \subset \partial\Omega$ satisfies that $F \subset \overline{\Gamma}$ or $F \subset \overline{\partial\Omega \setminus \Gamma}$ (i.e. Γ is union of faces of triangulation) then in all examples where $X_h \subset H^1(\Omega)$ we also have that $V_h \subset V$

(not allowed: Ω )

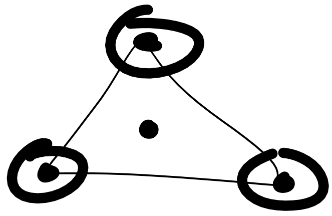
Example (Lagrange finite element): $X_h \subset C(\overline{\Omega})$ constructed using Lagrange finite elements. Then, $\forall v_h \in X_h$ and $\forall$ face F, $v_h|_F$ uniquely determined by values at nodes lying on F. Thus if $F \subset \Gamma$, then

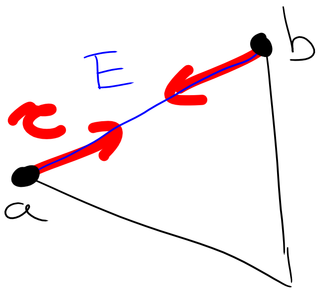
$v_h|_{\Gamma} = 0 \quad \forall v_h \in V_h$. Therefore, $v_h|_{\Gamma} = 0 \quad \forall v_h \in V_h$

$\Longrightarrow V_h \subset V$

<u>Example</u> (Cubic Hermite triangles)

Let $E \subset \Gamma$ be an edge of $T \in \mathcal{T}_h$.





For any $v_h \in V_h$

$$v_h(a) = v_h(b) = \frac{\partial v_h}{\partial \tau}(a) = \frac{\partial v_h}{\partial \tau}(b) = 0$$

Since $v_h|_E \in P_3(E) \Rightarrow v_h|_E = 0$

$\Longrightarrow V_h \subset V$.