Numerical Integration

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω & $\{X_h\}$ family of finite element spaces under same assumptions as for convergence of discrete solution.

For simplicity assume all FEs (T, P_T, Σ_T), $T \in \bigcup \mathcal{T}_h$, are affine-equivalent to one reference element $(\hat{T}, \hat{P}, \hat{\Sigma})$.

Introduce quadrature formula on reference element

$$(22) \quad \sum_{l=1}^{L} \hat{\omega}_l \hat{\varphi}(\hat{b}_l) \sim \int_{\hat{T}} \hat{\varphi} \, d\hat{x}$$

where $\hat{\omega}_l$ are positive real numbers called weights and $\hat{b}_l \in \hat{T}$ are points called nodes.

For any $T \in \bigcup \mathcal{T}_h$ $\exists$ invertible affine mapping $F_T(\hat{x}) = B_T \hat{x} + b_T$ such that $F_T(\hat{T}) = T$ & hence

$$\int_T \varphi \, dx = |\det B| \int_{\hat{T}} \hat{\varphi} \, dx \qquad \hat{\varphi} := \varphi \circ F_T$$

$\Rightarrow$ quadrature formula on T

$$\sum_{l=1}^{L} \omega_{l,T} \varphi(b_{l,T}) \sim \int_T \varphi \, dx$$

where $\omega_{l,T} = |\det B_T| \omega_l$, $b_{l,T} = F_T(\hat{b}_l)$, $l = 1, \dots, L$.

Introduce error of quadrature formula:

$$E_T(\varphi) = \int_T \varphi \, dx - \sum_{l=1}^{L} \omega_{l,T} \varphi(b_{l,T})$$

$$\hat{E}(\hat{\varphi}) = \int_{\hat{T}} \hat{\varphi} \, dx - \sum_{l=1}^{L} \hat{\omega}_l \hat{\varphi}(\hat{b}_l)$$

Clearly $E_T(\varphi) = |\det B|\, \hat{E}(\hat{\varphi})$

For any $k \in \mathbb{N}$ $\exists$ nodes & weights s.t.

$$\hat{E}(\hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in P_k(\hat{T})$$

Let us assume AVP is weak formulation of BVP

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $a_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ are defined everywhere in $\bar{\Omega}$.

Thus, $V = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$

$$\langle f, v \rangle = \int_\Omega f v \, dx$$

Assume that $\exists\, \theta > 0$ such that

(23) $$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall x \in \bar{\Omega},\ \xi \in \mathbb{R}^n$$

Then, assumptions of AVP are satisfied.

Set $V_h = \{ v_h \in X_h : \phi(v_h) = 0 \ \forall \phi \in \Sigma_h^{\partial\Omega} \}$

& assume that $V_h \subset H_0^1(\Omega) \cap C(\bar{\Omega})$.

We approximate the integrals over T:

(24) $$\begin{cases} a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,T} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j}\right)(b_{l,T}) \\ \langle f_h, v_h \rangle = \sum_{T \in \mathcal{T}_h} \sum_{l=1}^{L} \omega_{l,T} (f v_h)(b_{l,T}) \end{cases}$$

Here, $\left(a_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j}\right)(b_{l,T}) = \left[a_{ij} \frac{\partial}{\partial x_i}(u_h|_T) \frac{\partial}{\partial x_j}(v_h|_T)\right](b_{l,T})$

Discrete problem: Find $u_h \in V_h$ s.t. $a_h(u_h, v_h) = \langle f_h, v_h \rangle \ \forall v_h \in V_h$

Note a_h & f_h not defined on V from AVP

Discrete problem solveable $\Longleftrightarrow$ a_h is V_h-elliptic

$$\exists \tilde{\alpha} > 0 \text{ s.t. } a_h(v_h, v_h) \geq \tilde{\alpha} |v_h|_{1,\Omega}^2 \quad \forall v_h \in V_h$$

Theorem 11 (First Strang Lemma)

Let $u \in V$ be the solution of the AVP and let $u_h \in V_h$ satisfy $a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h$

where V_h is a subspace of V, $a_h : V_h \times V_h \to \mathbb{R}$ is a bilinear form, which is V_h-elliptic; i.e.,

$$\exists \tilde{\alpha} > 0 \text{ s.t. } a_h(v_h, v_h) \geq \tilde{\alpha} \|v_h\|_V^2 \quad \forall v_h \in V_h$$

and f_h is a linear functional on V_h. Then,

$$\|u - u_h\|_V \leq \frac{1}{\tilde{\alpha}} \inf_{v_h \in V_h} \Big\{ (M + \tilde{\alpha}) \|u - v_h\|_V + \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V} \Big\}$$

$$+ \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{|\langle f, w_h \rangle - \langle f_h, w_h \rangle|}{\|w_h\|_V}$$

consistency errors

Proof $\|u_h - v_h\|_V \leq \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{a_h(u_h - v_h, w_h)}{\|w_h\|_V}$

For $v_h, w_h \in V_h$

$$\begin{aligned} a_h(u_h - v_h, w_h) &= \langle f_h, w_h \rangle - a_h(v_h, w_h) \\ &= \langle f_h, w_h \rangle - \langle f, w_h \rangle + a(u, w_h) - a_h(v_h, w_h) \\ &= a(u - v_h, w_h) + a(v_h, w_h) - a_h(v_h, w_h) \\ &\quad + \langle f_h, w_h \rangle - \langle f, w_h \rangle \end{aligned}$$

From fact that $a(u - v_h, w_h) \leq M \|u - v_h\|_V \|w_h\|_V$ we have that

$$\|u - v_h\| \le \frac{M}{\tilde{\alpha}} \|u - v_h\|_V + \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V}$$
$$+ \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{|\langle f_h, w_h\rangle - \langle f, w_h\rangle|}{\|w_h\|_V}$$

Additionally, $\|u - u_h\| \le \|u - u_h\| + \|u_h - v_h\|$ $\square$

Theorem 12 Consider quadrature formula (22) defined on reference element $(\hat{T}, \hat{P}, \hat{\Sigma})$ and let $m \in \mathbb{N}$ such that $\hat{P} \subset P_m(\hat{T})$ and let there hold at least one of the following conditions:

(26) $\bigcup_{l=1}^{L} \{\hat{b}_l\}$ contains a $P_{m-1}(\hat{T})$-unisolvent set

(27) $\hat{E}(\hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in P_{2m-2}(\hat{T})$

Then, the bilinear form a_h in defined in (24) is uniformly V_h-elliptic; i.e, $\exists \tilde{\alpha} > 0$, independent of h, s.t
$$a_h(v_h, v_h) \ge \tilde{\alpha} |v_h|^2_{1,\Omega} \quad \forall v_h \in V_h, \forall h > 0$$

Theorem 13 (Bramble-Hilbert Lemma)
Let $G \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Let $k \in \mathbb{N}_0, p \in [1, \infty]$ and let $l \in [W^{k+1,p}(G)]'$ satisfying $l(q) = 0 \quad \forall q \in P_k(G)$.
Then, there is a constant C depending only on G, k, p such that
$$|l(v)| \le C \|l\|_{[W^{k+1,p}(G)]'} |v|_{k+1,p,G} \quad \forall v \in W^{k+1,p}(G)$$

Proof For any $v \in W^{k+1,p}(G)$
$$|l(v)| = |l(v+q)| \le \|l\|_{[W^{k+1,p}(G)]'} \|v+q\|_{k+1,p,G} \quad \forall q \in P_k(G)$$

$$\Rightarrow |\ell(v)| \le \|\ell\|_{[W^{k+1,p}(G)]'} \inf_{q \in P_k(G)} \|v+q\|_{k+1,p,G}$$

$$\underset{(\text{Thm } 1)}{\le} C \|\ell\|_{[W^{k+1,p}(G)]'} |v|_{k+1,p,G} \qquad \square$$

Theorem 14 Let $G \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary, $m \in \mathbb{N}_0$ and $p \in [1,\infty]$. Let $\varphi \in W^{m,p}(G)$ and $\omega \in W^{m,\infty}(G)$. Then, $\varphi\omega \in W^{m,p}(G)$ and

(29) $$|\varphi\omega|_{m,p,G} \le C \sum_{j=1}^{m} |\varphi|_{m-j,p,G} |\omega|_{j,\infty,G}$$

where C depends only on m & n.

Proof From Liebniz formula. (skip)

Theorem 15 Let $k \in \mathbb{N}$ be such that

$$\hat{E}(\hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in P_{2k-2}(\hat{T}).$$

Then, there exists a constant C, independent of $T \in \mathcal{T}_h$ and h such that

(30) $$|E_T(auv)| \le C h_T^k \|a\|_{k,\infty,T} \|u\|_{k-1,T} \|v\|_{0,T}$$
$$\forall a \in W^{k,\infty}(T),\ u,v \in P_{k-1}(T).$$

Proof $E_T(auv) = |\det B_T| \hat{E}(\hat{a}\hat{u}\hat{v})$, $\hat{a} \in W^{k,\infty}(\hat{T})$

For any $\hat{w} \in P_{k-1}(\hat{T})$ and $\hat{\varphi} \in W^{k,\infty}(\hat{T})$

$$|\hat{E}(\hat{\varphi}\hat{w})| = \left| \int_{\hat{T}} \hat{\varphi}\hat{w}\,dx - \sum_{\ell=1}^{L} \hat{\omega}_\ell (\hat{\varphi}\hat{w})(\hat{b}_\ell) \right| \le \hat{C}_1 \|\hat{\varphi}\|_{0,\infty,\hat{T}} \|\hat{w}\|_{0,\infty,\hat{T}}$$

$$\le \hat{C}_2 \|\hat{\varphi}\|_{k,\infty,\hat{T}} \|\hat{w}\|_{0,\hat{T}} \quad \text{(by equiv. of norms)}$$

For any $\hat{w} \in P_{k-1}(\hat{T})$ $\hat{E}(\cdot\,\hat{w}) \in [W^{k,\infty}(\hat{T})]'$,

$$\|\hat{E}(\cdot\,\hat{w})\|_{[W^{k,\infty}(\hat{T})]'} \le \hat{C}_2 \|\hat{w}\|_{0,\hat{T}} \text{ and } \hat{E}(\hat{\varphi}\hat{w}) = 0 \ \forall \hat{\varphi} \in P_{k-1}(\hat{T})$$

$$\underset{\text{Thm 13}}{\Longrightarrow} \quad |\hat{E}(\hat{\varphi}\hat{w})| \le \hat{C}_3 |\hat{\varphi}|_{k,\infty,\hat{T}} \|\hat{w}\|_{0,\hat{T}} \qquad \forall \hat{\varphi} \in W^{k,\infty}(\hat{T}),\ \hat{w} \in P_{k-1}(\hat{T})$$

$$\Longrightarrow |\hat{E}(\underbrace{\hat{a}\hat{u}}_{\hat{\varphi}}\underbrace{\hat{v}}_{\hat{w}})| \le \hat{C}_3 |\hat{a}\hat{u}|_{k,\infty,\hat{T}} \|\hat{v}\|_{0,\hat{T}}$$

$$\le \hat{C}_4 \sum_{j=0}^{k} |\hat{a}|_{k-j,\infty,\hat{T}} |\hat{u}|_{j,\infty,\hat{T}} \|\hat{v}\|_{0,\hat{T}}$$

$$\le \hat{C}_5 \sum_{j=0}^{k-1} |\hat{a}|_{k-j,\infty,\hat{T}} |\hat{u}|_{j,\hat{T}} \|\hat{v}\|_{0,\hat{T}}$$

By Thm 2 & 3:

$$|\hat{a}|_{k-j,\infty,\hat{T}} \le C(k,n) \|B_T\|^{k-j} |a|_{k-j,\infty,T}$$

$$\le \hat{C}_6 h_T^{k-j} |a|_{k-j,\infty,T} \qquad j=0,\dots,k-1$$

$$|\hat{u}|_{j,\hat{T}} \le C(k,n) \|B_T\|^j |\det B_T|^{-1/2} |u|_{j,T}$$

$$\le \hat{C}_6 h_T^j |\det B_T|^{-1/2} |u|_{j,T} \qquad j=0,\dots,k-1$$

$$\|\hat{v}\|_{0,\hat{T}} \le C(n) |\det B_T|^{-1/2} \|v\|_{0,T}$$

$$\Longrightarrow |E_T(auv)| \le \hat{C}_7 \sum_{j=0}^{k-1} h_T^k |a|_{k-j,\infty,T} |u|_{j,T} \|v\|_{0,T}$$

$$\le \hat{C}_8 h_T^k \|a\|_{k,\infty,T} \|u\|_{k-1,T} \|v\|_{0,T} \qquad \square$$

<u>Theorem 16</u> Let $k \in \mathbb{N}$ be such that

$$\hat{E}(\hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in P_{2k-2}(\hat{T})$$

and let $q \in [1,\infty]$ satisfy $k > \frac{n}{q}$ $\left(\Rightarrow W^{k,q}(\hat{T}) \hookrightarrow C(\hat{T})\right)$.

Then, there is a constant C independent of $T \in \mathcal{T}_h$ and h such that

$$(31) \quad |E_T(fp)| \le C h_T^k |T|^{\frac{1}{2}-\frac{1}{q}} \|f\|_{k,q,T} \|p\|_{1,T}$$

$$\forall f \in W^{k,q}(T),\ p \in P_k(T).$$