Theorem 17 Let all assumptions for Theorem 7 be satisfied and, additionally, assume there is only one reference element $(\hat{T}, \hat{P}, \hat{\Sigma})$ and it holds that

$$\hat{P} = P_k(\hat{T})$$

Let the quadrature formula (22) be defined on $\hat{T}$ and let

$$\hat{E}(\hat{\varphi}) = 0 \quad \forall \hat{\varphi} \in P_{2k-2}(\hat{T})$$

Consider the variational problem introduced above with $a_{ij} \in W^{k,\infty}(\Omega)$, $i,j = 1, \dots, n$ and $f \in W^{k,q}(\Omega)$, where $q \in [2, \infty]$ and $k > \frac{n}{q}$. Let the weak solution of this VP satisfy $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Then, there exists a constant C independent of h such that the solution of the discrete problem (25) satisfies

$$\|u - u_h\|_{1,\Omega} \le C h^k \Big\{ |u|_{k+1,\Omega} + \sum_{i,j=1} \|a_{ij}\|_{k,\infty,\Omega} \|u\|_{k+1,\Omega} + \|f\|_{k,q,\Omega} \Big\}$$

Proof

(30) $$|E_T(auv)| \le C h_T^k \|a\|_{k,\infty,T} \|u\|_{k-1,T} \|v\|_{0,T} \quad \forall a \in W^{k,\infty}(T),\ u, v \in P_{k-1}(T)$$

(31) $$|E_T(fp)| \le C h_T^k |T|^{1/2 - 1/q} \|f\|_{k,q,T} \|p\|_{1,T} \quad \forall f \in W^{k,q}(T),\ p \in P_k(T)$$

According to theorems 11 & 12

$$\|u - u_h\|_{1,\Omega} \le C \Big[\|u - \Pi_h u\|_{1,\Omega} + \sup_{w_h \in V_h} \frac{|a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)|}{\|w_h\|_{1,\Omega}} + \sup_{w_h \in V_h} \frac{|(f, w_h) - (f_h, w_h)|}{\|w_h\|_{1,\Omega}} \Big]$$

According to the assumptions

$$\|u-\Pi_h u\|_{1,\Omega} \le C h^k |u|_{k+1,\Omega} \qquad \text{(Thm. 6)}$$

Furthermore,

$$|a(\Pi_h u, w_h) - a_h(\Pi_h u, w_h)|$$

$$\le \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{n} \left| E_T\left(a_{ij} \frac{\partial}{\partial x_i}(\Pi_h u|_T) \frac{\partial}{\partial x_j}(w_h|_T)\right)\right|$$

$$\le C h^k \sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{n} \|a_{ij}\|_{k,\infty,T} \|\Pi_h u\|_{k,T} |w_h|_{1,T}$$

$$\le C h^k \left(\sum_{i,j=1}^{n} \|a_{ij}\|_{k,\infty,\Omega}\right) \underbrace{\|\Pi_h u\|_{k,h}}_{\le \tilde{C}\|u\|_{k+1,\Omega}} |w_h|_{1,\Omega}$$

$$\|\Pi_h u\|_{k,h} \le \|\Pi_h u - u\|_{k,h} + \|u\|_{k,\Omega} \le C h |u|_{k+1,\Omega} + \|u\|_{k,\Omega}$$

$$\le \tilde{C} \|u\|_{k+1,\Omega}$$

Finally,

$$|\langle f, w_h\rangle - \langle f, w_h\rangle| \le \sum_{T\in\mathcal{T}_h} |E_T(f w_h)|$$

$$\le \sum_{T\in\mathcal{T}_h} C h_T^k |T|^{1/2-1/q} \|f\|_{k,q,T} \|w_h\|_{1,T} \quad \text{(by (31))}$$

$$\sum_T |a_T b_T c_T| \le \left(\sum_T |a_T|^r\right)^{1/r} \left(\sum_T |b_T|^s\right)^{1/s} \left(\sum_T |c_T|^t\right)^{1/t}$$

$$\forall r,s,t \ge 1, \ \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$$

Set $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$, $s=q$, $t=2$. Then,

$$|\langle f, w_h\rangle - \langle f_h, w_h\rangle| \le C h^k |\Omega|^{1/2-1/q} \|f\|_{k,q,\Omega} \|w_h\|_{1,\Omega} \qquad \square$$

FE for parabolic problems

$\Omega \subset \mathbb{R}^n$ bounded, Lipschitz-continuous boundary,
$\Omega_T := \Omega \times [0,T]$, where $T>0$ is a fixed time.
Consider a problem for $u = u(x,t)$

$$(32) \begin{cases} \frac{\partial u}{\partial t} + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases}$$

where L is an elliptic linear differential operator of 2nd order (w.r.t x). For simplicity, it is assumed that L <u>does not</u> depend on t.

The function u is a weak solution of (32) if

$$u \in L^2(0,T,H_0^1(\Omega)),\ u' \in L^2(0,T,H^{-1}(\Omega))$$

$\left(\frac{\partial u}{\partial t}\right)$

$$\langle u', v\rangle + a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega) \text{ and a.e. } t \in [0,T]$$

$(u:[0,T] \to H_0^1(\Omega))$

where $a: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bilinear form corresponding to L, $L^2(0,T,X)$ is the Bochner space

$$\left(\int_0^T \|u(t)\|_X^2 \, d\mu\right)^{1/2} < +\infty \quad \& \ H^{-1}(\Omega) \equiv [H_0^1(\Omega)]'.$$

We assume a satisfies the assumptions of the AVP.
Then, $\forall f \in L^2(\Omega_T)$ and $\forall u_0 \in L^2(\Omega)$ $\exists!$ weak solution of (32).

We also assume that

$$(33) \quad |a(v,w) - a(w,v)| \le K |v|_{1,\Omega} \|w\|_{0,\Omega} \quad \forall v,w \in H_0^1(\Omega)$$

For $L = -\sum_{i,j}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu$

Then $a(u,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} v + cuv \, dx$

Semidiscretisation in Space (Method of Lines)

Let $\{V_h\}$ be family of finite element spaces on Ω defined like above such that $V_h \subset H_0^1(\Omega)$.

For any k, we define the discrete solution

$u_h : [0,T] \to V_h$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,\ t \in (0,T],$$

$$u_h(0) = u_{0h}$$

where $u_{0h} \in V_h$ is an approximation of u_0

Let $\{p_i\}_{i=1}^{N_h}$ be the basis of V_h. Then,

$$u_h(x,t) = \sum_{j=1}^{N_h} u_j(t) p_j(x)$$

$$u_{0h}(x) := \sum_{j=1}^{N_h} \gamma_j p_j(x).$$

Then, for any $t \in (0,T]$,

$$\sum_{j=1}^{N_h} (p_j, p_i) u_j'(t) + \sum_{j=1}^{N_h} a(p_j, p_i) u_j(t) = (f, p_i) \quad i = 1, \dots, N_h$$

$$u_j(0) = \gamma_j \quad j = 1, \dots, N_h$$

Denote $M = (m_{ij})_{i,j=1}^{N_h}$, $m_{ij} = (p_j, p_i)$ (mass matrix)

$A = (a_{ij})_{i,j=1}^{N_h}$, $a_{ij} = a(p_j, p_i)$ (stiffness matrix)

$F = (f_i)_{i=1}^{N_h}$, $f_i = (f, p_i)$

$U = (u_i)_{i=1}^{N_h}$, $U_0 = (\gamma_i)_{i=1}^{N_h}$

Then, the discrete problem can be written equivalently as

$$\mathbb{M}U'(t) + \mathbb{A}U(t) = F(t), \quad t \in (0,T]$$
$$U(0) = U_0$$

This is a system of N_h linear <u>ordinary</u> differential equs.

Since $\mathbb{M}$ is SPD ($\Rightarrow$ non-singular), we have

$$U'(t) + \mathbb{M}^{-1}\mathbb{A}U(t) = \mathbb{M}^{-1}F(t), \quad t \in (0,T]$$
$$U(0) = U_0$$

This system has a unique solution such that the equation is satisfied for a.a. $t \in [0,T]$. If F is continuous on $[0,T]$, then U' is continuous on $[0,T]$ and hence U satisfies the equation $\forall t \in [0,T]$.

This will always be assumed in the following.

<u>Theorem 18</u> Let all assumptions used for Theorems 7 & 10 be satisfied. In particular, let $P_k(\hat{T}_i) \subset \hat{P}_i$, $i = 1,\dots,M$. Let $u_0 \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$

$$u, u_t \in L^2(0,T, H^{k+1}(\Omega) \cap H_0^1(\Omega)).$$

Then,

(A) $$\|u(t) - u_h(t)\|_{0,\Omega} \le \|u_0 - u_{0h}\|_{0,\Omega} + Ch^{k+1}\left(|u_0|_{k+1,\Omega} + \int_0^t |u_t(s)|_{k+1,\Omega}\,ds\right)$$

(B) $$|u(t) - u_h(t)|_{1,\Omega} \le Ch^k |u(t)|_{k+1,\Omega} + C\left(e^{\frac{K^2 t}{4\alpha}}\right)\left[|u_0 - u_{0h}|_{1,\Omega} + h^k\left(|u_0|_{k+1,\Omega} + \left(\int_0^t |u_t(s)|_{k,\Omega}^2\,ds\right)^{1/2}\right)\right]$$

$$\forall t \in [0,T].$$