Theorem 18 Let all assumptions used for Theorems 7 & 10 be satisfied. In particular, let $P_k(\hat{T}_i) \subset \hat{P}_i, i=1,\dots,M$. Let $u_0 \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$

$u, u_t \in L^2(0,T, H^{k+1}(\Omega) \cap H_0^1(\Omega))$.

Then,

(A) $$\|u(t)-u_h(t)\|_{0,\Omega} \le \|u_0-u_{0h}\|_{0,\Omega} + Ch^{k+1}\left(|u_0|_{k+1,\Omega} + \int_0^t |u_t(s)|_{k+1,\Omega}\, ds\right)$$

(B) $$|u(t)-u_h(t)|_{1,\Omega} \le Ch^k |u(t)|_{k+1,\Omega} + C\left(e^{\frac{|c|^2 t}{4\alpha}}\right)\left[|u_0-u_{0h}|_{1,\Omega} + h^k\left(|u_0|_{k+1,\Omega} + \left(\int_0^t |u_t(s)|^2_{k,\Omega}\, ds\right)^{1/2}\right)\right]$$
$\forall t \in [0,T]$.

Proof Define the operator $R_h : H_0^1(\Omega) \to V_h$ by
$$a(R_h v, v_h) = a(v, v_h) \quad \forall v \in H_0^1(\Omega), v_h \in V_h$$

- R_h is a projection.

$R_h v$ is a discrete solution of the AVP with exact solution v. Thus,
$$\|v - R_h v\|_{1,\Omega} \le Ch^k |v|_{k+1,\Omega}$$
$$\|v - R_h v\|_{0,\Omega} \le Ch^{k+1} |v|_{k+1,\Omega} \quad \forall v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$$

Denote $\rho = u - R_h u$, $\Theta = u_h - R_h u$ $(\rho(t), \Theta(t))$

Then, $u - u_h = \rho - \Theta$

$$\|\rho(t)\|_{0,\Omega} \le Ch^{k+1} |u(t)|_{k+1,\Omega}$$
$$= Ch^{k+1} \left| u_0 + \int_0^t (u_t)(s)\, ds \right|_{k+1,\Omega}$$
$$\le Ch^{k+1}\left(|u_0|_{k+1,\Omega} + \int_0^t |u_t(s)|_{k+1,\Omega}\, ds\right) \quad (C)$$

$$\left[\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) = (f, v_h)\right]$$

$$(\ast)\quad (\theta_t, v_h) + a(\theta, v_h)$$

$$= \underbrace{\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h)}_{(f,v_h)=(u_t,v_h)+a(u_h,v_h)} - \left(\frac{\partial}{\partial t} R_h u_h, v_h\right) - \underbrace{a(R_h u, v_h)}_{a(u,v_h)}$$

$$= \left(\frac{\partial}{\partial t}(u - R_h u), v_h\right) = (\rho_t, v_h) \quad \forall v_h \in V_h,\ t \in [0,T]$$

For $v_h = \theta$

$$\underbrace{(\theta_t, \theta)}_{\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{0,\Omega}} + \underbrace{a(\theta, \theta)}_{\geq 0} = (\rho_t, \theta) \leq \|\rho_t\|_{0,\Omega} \|\theta\|_{0,\Omega}$$

Ideally, want to do:

$$\cancel{\|\theta\|_{0,\Omega}} \frac{d}{dt}\|\theta\|_{0,\Omega} \leq \|\rho_t\|_{0,\Omega} \cancel{\|\theta\|_{0,\Omega}}$$

However, $\|\theta\|_{0,\Omega}$ may not be differentiable w.r.t. t.

E.g. for $\theta(t) = t - c$, we have that

$\|\theta(t)\|_{0,\Omega} = \sqrt{|\Omega|}\,|t-c|$ – not differentiable at $t = c$.

For $\varepsilon > 0$ arbitrary

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{0,\Omega} = \frac{1}{2}\frac{d}{dt}\left[\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2}\right]^2$$

$$= \left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2} \frac{d}{dt}\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2}$$

$$\Rightarrow \cancel{\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2}} \frac{d}{dt}\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2}$$

$$\leq \|\rho_t\|_{0,\Omega} \cancel{\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2}}$$

$$\Rightarrow \frac{d}{dt}\left(\|\theta\|^2_{0,\Omega} + \varepsilon^2\right)^{1/2} \leq \|\rho_t\|_{0,\Omega}$$

$$\Rightarrow \left(\|\theta(t)\|_{0,\Omega}^2+\varepsilon^2\right)^{1/2} - \left(\|\theta(0)\|_{0,\Omega}^2+\varepsilon^2\right)^{1/2} \le \int_0^t \|\rho_t(s)\|_{0,\Omega}\,ds$$

(Int. over $\int_0^t$)

$$\overset{\varepsilon\to 0}{\Longrightarrow} \|\theta(t)\|_{0,\Omega} \le \|\theta(0)\|_{0,\Omega} + \int_0^t \|\rho_t(s)\|_{0,\Omega}\,ds \qquad (D)$$

$$\|\theta(0)\|_{0,\Omega} = \|u_{0h}-R_h u_0\|_{0,\Omega}$$
$$\le \|u_{0h}-u_0\|_{0,\Omega} + \|u_0 - R_h u_0\|_{0,\Omega}$$
$$\le \|u_{0h}-u_0\|_{0,\Omega} + Ch^{k+1}|u_0|_{k+1,\Omega} \qquad (E)$$

$$\|\rho_t\|_{0,\Omega} = \|(u-R_h u)_t\|_{0,\Omega} = \|u_t - R_h u_t\|_{0,\Omega} \le Ch^{k+1}|u_t|_{k+1,\Omega} \qquad (F)$$

Combine (C)–(F) gives (A)

Setting $v_h = \theta_t$ in ($\ast$)

$$\|\theta_t\|_{0,\Omega}^2 + a(\theta,\theta_t) = (\rho_t,\theta_t) \le \|\rho_t\|_{0,\Omega}\|\theta_t\|_{0,\Omega}$$
$$\le \frac{1}{2}\|\rho_t\|_{0,\Omega}^2 + \frac{1}{2}\|\theta_t\|_{0,\Omega}^2 \qquad (G)$$

(Young's Ineq.)

$$\frac{d}{dt}a(\theta,\theta) = a(\theta_t,\theta) + a(\theta,\theta_t)$$
$$= 2a(\theta,\theta_t) + \underbrace{a(\theta_t,\theta) - a(\theta,\theta_t)}_{\le K\|\theta_t\|_{0,\Omega}|\theta|_{1,\Omega}}$$

$$\le 2a(\theta,\theta_t) + \|\theta_t\|_{0,\Omega}^2 + \frac{K^2}{4}|\theta|_{1,\Omega}^2 \quad \text{(Young's)}$$
$$\le \|\rho_t\|_{0,\Omega}^2 + \frac{K^2}{4\alpha}a(\theta,\theta) \quad \text{(by (G) \& V-elliptic)}$$

Gronwell's Lemma $\Rightarrow$

$$a(\theta(t),\theta(t)) \le e^{\frac{K^2 t}{4\alpha}}\left[a(\theta(0),\theta(0)) + \int_0^t \|\rho_t(s)\|_{0,\Omega}^2\,ds\right] \quad \forall t\in[0,T].$$

$$\Rightarrow |\theta(t)|_{1,\Omega}^2 \le \frac{1}{\alpha} e^{\frac{k^2 t}{4\alpha}} \left[M |\theta(0)|_{1,\Omega}^2 + \int_0^t \|\rho_t(s)\|_{0,\Omega}^2 \, ds \right]$$

Combine results gives (B) $\square$

Discretisation in time

For simplicity: backward Euler with timestep $\tau > 0$:

$t_j = j\tau, \; k = 0, 1, 2, \ldots$

$u(\cdot, t_j) \approx u_h^j \in V_h$

$$\frac{\partial u_h}{\partial t}(\cdot, t_k) \approx \frac{u_h^k - u_h^{k-1}}{\tau}$$

$\Rightarrow$ Obtain ~~fully discrete~~ fully discrete problem:

find $u_h^j \in V_h$, $j = 1, 2, 3 \ldots$, such that

$$\left(\frac{u_h^j - u^{j-1}}{\tau}, v_h \right) + a(u^j, v_h) = (f(t_j), v_h) \quad \forall v_h \in V_h, \; j \ge 1$$

$$u_h^0 = u_{0h}$$

If u_h^{j-1} is given, then u_h^k simply determined by

$$(u_h^j, v_h) + \tau a(u_h^j, v_h) = (u_h^{j-1} + \tau f(t_j), v_h) \quad \forall v_h \in V_h$$

which is a FE discretisation of $(I + \tau L)u = g$

In matrix form:

$$\underbrace{(M + \tau A)}_{\text{positive def.} \Rightarrow \text{non-sing.}} U_h^j = M U_h^{j-1} + \tau F(t_j)$$

Theorem 19 Let assumptions of Theorem 18 hold, and additionally, $u_{tt} \in L^2(0,T, L^2(\Omega))$. Then,

$$\|u(t_j) - w^j_h\|_{0,\Omega} \le \|u_0 - u_{0h}\|_{0,\Omega} + Ch^{k+1}\left(|u_0|_{k+1,\Omega} + \int_0^{t_j} |u_t(s)|_{k+1,\Omega}\,ds\right) + \tau \int_0^{t_j} \|u_{tt}(s)\|_{0,\Omega}\,ds$$

$$|u(t_j) - w^j_h|_{1,\Omega} \le C\sqrt{e^{\frac{K^2 t_k}{4\alpha}}}\left[|u_0 - u_{0h}|_{1,\Omega} + h^k\left(|u_0|_{k+1,\Omega} + \left(\int_0^{t_j} |u_t(s)|^2_{k,\Omega}\,ds\right)^{1/2}\right) + \tau\left(\int_0^{t_j} \|u_{tt}(s)\|^2_{0,\Omega}\,ds\right)^{1/2}\right] + Ch^k |u(t_j)|_{k+1,\Omega}$$