Weak formulation of general 2nd-order elliptic PDE

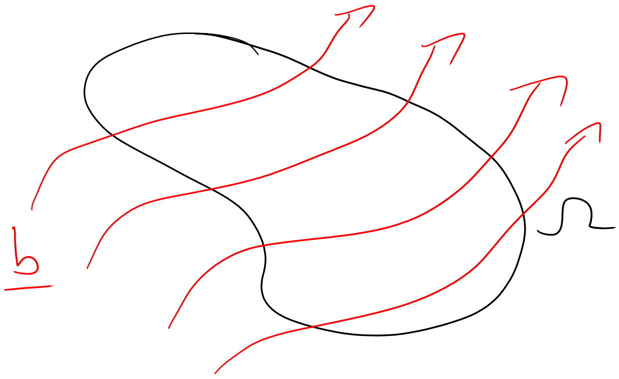
$$\underbrace{-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)}_{\text{diffusion}} + \underbrace{\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}}_{\text{convection/advection}} + \underbrace{cu}_{\text{reaction}} = f \quad \text{in } \Omega \subset \mathbb{R}^n$$

forcing function

bounded domain with Lipschitz continuous boundary.

$\underline{b} = (b_1, \dots, b_n)$ – convective vector (velocity field)

(depends on x)



$$-\sum_{j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j} = h_i \qquad \underline{h} = (h_1, \dots, h_n) \text{ – flux of } u$$

depends on x

Define $A = (a_{ij})_{i,j=1}^{n}$, $h = A\nabla u$ heat/mass flux

when $A = aI$ isotropic material – diffusion same in all directions

otherwise anisotropic

If $a = \text{const} \to$ homogeneous material – same at all part

otherwise inhomogeneous

Can write eqn as

$$-\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu = f$$

Boundary conditions

- value on boundary - Dirichlet BCs $u = u_b$ on $\partial\Omega$ (e.g. temperature on boundary)

- Neumann B.Cs : $h \cdot n = g$ on $\partial\Omega$ - normal flux on boundary

$$-\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i = g$$

For simplicity start with homogeneous Dirichlet BCs:

$$u = 0 \text{ on } \partial\Omega$$

Assume:

- $a_{ij}, b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$
- A uniformly positive definite:

$$\exists \alpha > 0 : \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^n \text{ a.e in } \Omega$$

$\Rightarrow$ Elliptic PDE

Multiply by $v \in C_0^\infty(\Omega)$ and integrate

$$-\underbrace{\sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) v \, dx}_{\underbrace{\int_\Omega \frac{\partial}{\partial x_i}\left[a_{ij} \frac{\partial u}{\partial x_j} v\right]}_{\text{Gauss Integral Theorem}} - \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx} + \int_\Omega b \cdot \nabla v \, dx + \int_\Omega c u v \, dx = \int_\Omega f v \, dx$$

$$\int_{\partial\Omega} \underbrace{n_i a_{ij} \frac{\partial u}{\partial x_j}}_{\text{Neumann}} v \, ds = 0$$

$\hookrightarrow = 0$ (Dirichlet, $v \in C_0^\infty(\Omega)$)

$\Rightarrow u$ satisfies $a(u,v) = \langle F, v\rangle$ where

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \int_\Omega b \cdot \nabla u\, v\, dx + \int_\Omega c u v\, dx$$

$$\langle F, v\rangle = \int_\Omega f v\, dx$$

Denote by $V = H_0^1(\Omega)$, then $u \in V$ weak solution if

$$a(u,v) = \langle F, v\rangle \quad \forall v \in V$$

<u>Remark</u> If $u = u_b$ on Γ_D, $-\sum_{i,j=1}^{n} n_i a_{ij} \frac{\partial u}{\partial x_j} = g$ on Γ_N

where $\partial\Omega = \Gamma_D \cap \Gamma_N$. Then,

$$V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}$$

<u>Prove Continuity</u>

- $|\langle F, v\rangle| = \left|\int_\Omega f v\, dx\right| \leq \|f\|_{0,\Omega} \|v\|_{0,\Omega}$ (Hölder's)

$$\leq \|f\|_{0,\Omega} \|v\|_{1,\Omega} \quad \text{(Norm or semi-norm)}$$

- $$|a(u,v)| \leq \sum_{i,j=1}^{n} \|\underset{\in L^\infty}{a_{ij}}\|_{0,\infty,\Omega} \int_\Omega \left|\frac{\partial u}{\partial x_j}\right| \left|\frac{\partial v}{\partial x_i}\right| dx$$

$$+ \sum_{i=1}^{n} \|\underset{\in L^\infty}{b_i}\|_{0,\infty,\Omega} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right| |v|\, dx$$

$$+ \|\underset{\in L^\infty}{c}\|_{0,\infty,\Omega} \int_\Omega |u||v|\, dx$$

$$= \left(\sum_{i,j=1}^{n} \|a_{ij}\|_{0,\infty,\Omega}^2\right)^{1/2} |u|_{1,\Omega} |v|_{1,\Omega} + \|b\|_{0,\infty,\Omega} |u|_{1,\Omega} \|v\|_{0,\Omega}$$

$$+ \|c\|_{0,\infty,\Omega} \|u\|_{0,\Omega} \|v\|_{0,\Omega}$$

$$\leq M \|u\|_{1,\Omega} \|v\|_{1,\Omega} \text{ where } M = \max(\text{constants})$$

Prove ellipticity : $a(u,u) \geq \alpha \|u\|_{1,\Omega}^2 \quad \forall u \in V$

$$a(u,u) = \underbrace{\sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dx}_{\geq \int_\Omega \alpha \|\nabla u\|^2 dx} + \sum_{i=1}^{n} \int_\Omega b_i \frac{\partial u}{\partial x_i} u \, dx + \int_\Omega c u^2 dx$$

$$\geq \int_\Omega \alpha \|\nabla u\|^2 dx = \alpha |u|_{1,\Omega}^2 \geq \bar{\alpha} \|u\|_{1,\Omega}^2 \quad (\text{if } \underline{b} \equiv 0 \ \& \ c \equiv 0)$$

- If $\underline{b} \equiv 0$
  - if $c \geq 0$ then $a(u,u) \geq \alpha |u|_{1,\Omega}^2 + \overbrace{\int_\Omega c u^2 dx}^{\geq 0}$
  $$\geq \bar{\alpha} \|u\|_{1,\Omega}^2$$
  - if $c < 0$ then OK if $|c|$ is sufficiently small (bounded by α).

- If $\underline{b} \neq 0$, $c = 0$

$$\left| \int_\Omega (\underline{b} \cdot \nabla u) u \, dx \right| \leq \|\underline{b}\|_{0,\infty,\Omega} |u|_{1,\Omega} \|u\|_{0,\Omega}$$
$$\leq C_0 \|\underline{b}\|_{0,\infty,\Omega} |u|_{1,\Omega}^2$$

If $C_0 \|\underline{b}\|_{0,\infty,\Omega} \leq \frac{\alpha}{2}$ then

$$a(u,u) \geq \alpha |u|_{1,\Omega}^2 - C_0 \|\underline{b}\|_{0,\infty,\Omega} |u|_{1,\Omega}^2$$
$$\geq \frac{\alpha}{2} |u|_{1,\Omega}^2$$

- Consider integration of $\underline{b}$ terms by parts:

$$\sum_{i=1}^{n} \int_\Omega b_i \underbrace{\frac{\partial u}{\partial x_i} u}_{\frac{1}{2} \frac{\partial}{\partial x_i} u^2} dx + \int_\Omega c u^2 dx = -\sum_{i=1}^{n} \int_\Omega \frac{\partial b_i}{\partial x_i} \frac{1}{2} u^2 dx + \int_\Omega c u^2 dx$$

$$= \int_\Omega \underbrace{\left(c - \tfrac{1}{2} \operatorname{div} \underline{b}\right)}_{\text{if } c - \frac{1}{2}\operatorname{div}\underline{b} \geq 0} u^2 dx$$

$\Rightarrow$ ellipticity.

Biharmonic Equation (4th Order)

$$\Delta^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

$$\left(\Delta^2 u = \Delta(\Delta u) = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^2}, \right.$$

$$\left. \nabla u \cdot n = \frac{\partial u}{\partial n} = \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} n_i \right)$$

From first Greens identity:

$$\int_\Omega \Delta u \cdot v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds$$

$$\int_\Omega u \cdot \Delta v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial n} u \, ds$$

$$\Rightarrow \int_\Omega (\Delta u) v - u (\Delta v) \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v - u \frac{\partial v}{\partial n} \, ds$$

(second Green's identity)

$\Rightarrow$ (Letting $u = \Delta u$):

$$\int_\Omega (\Delta^2 u) v \, dx = \int_\Omega (\Delta u)(\Delta v) \, dx + \int_{\partial\Omega} \frac{\partial}{\partial n} (\Delta u) v \, ds$$

$$- \int_{\partial\Omega} \Delta u \frac{\partial v}{\partial n} \, ds$$

Multiply PDE by $v \in C_0^\infty(\Omega)$, integrate & apply this identity noting that $v = 0$ & $\frac{\partial v}{\partial n} = 0$ as v has compact support:

$$\int_\Omega (\Delta u)(\Delta v) \, dx = \int_\Omega f v \, dx \quad \forall v \in C_0^\infty(\Omega)$$

Sufficient for u & v to have two weak derivatives

$\Rightarrow$ sufficient for $u, v \in H_0^2(\Omega)$

Recall $H_0^2(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{2,\Omega}} \subset H^2(\Omega)$

where $\|v\|_{2,\Omega}^2 = \|v\|_{0,\Omega}^2 + |v|_{1,\Omega}^2 + |v|_{2,\Omega}^2$

and $|v|_{2,\Omega}^2 = \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{0,\Omega}^2 + \left\|\frac{\partial^2 u}{\partial x \partial y}\right\|_{0,\Omega}^2 + \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{0,\Omega}^2$

If $\partial\Omega \in C^{1,1}$ then $u \in H_0^2(\Omega) \Rightarrow u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

If $\partial\Omega \in C^{2,1}$ then $H_0^2(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$

$\Rightarrow$ Weak formulation: Find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^2(\Omega)$$

Friedrichs inequality: $\|u\|_{2,\Omega} \leq C |u|_{2,\Omega} \quad \forall u \in H_0^2(\Omega)$

$\Rightarrow$ norm on $H^2(\Omega)$, can use $\|\cdot\|_{2,\Omega}$ or $|u|_{2,\Omega}$

Continuity:

$a(u,v) = \int_\Omega (\Delta u)(\Delta v) \, dx \qquad \langle F, v\rangle = \int f v \, dx$

Assume $f \in L^2(\Omega) \Rightarrow |\langle F, v\rangle| \leq \|f\|_{0,\Omega} \|v\|_{0,\Omega}$

$\leq C \|f\|_{0,\Omega} |v|_{2,\Omega} \quad \forall v \in H_0^2(\Omega)$

$\Rightarrow F$ continuous.

$|a(u,v)| \leq \|\Delta u\|_{0,\Omega} \|\Delta v\|_{0,\Omega}$

$\|\Delta u\|_{0,\Omega}^2 = \int_\Omega |\Delta u|^2 dx = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^2 dx\,dy$

$\leq 2 \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 dx\,dy \leq 2 |u|_{2,\Omega}^2$

$0 \leq (a-b)^2 = a^2 - 2ab + b^2 \Rightarrow 2ab \leq a^2 + b^2$

$(a+b)^2 = a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$

$\Rightarrow |a(u,v)| \leq 2 |u|_{2,\Omega} |v|_{2,\Omega} \Rightarrow a$ continuous.

Ellipticity Show $a(u,u) \geq \alpha |u|_{2,\Omega}^2$ $\forall u \in H_0^2(\Omega)$

$$a(u,u) = \int_\Omega |\Delta u|^2 dx = \|\Delta u\|_{0,\Omega}^2$$

← mixed derivative

← no mixed derivative

Let $u \in C_0^\infty(\Omega)$:

$$\|\Delta u\|_{0,\Omega}^2 = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^2$$

By integration by parts:

$$\int_\Omega \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\,dx\,dy = -\int_\Omega \frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x \partial y^2}\,dx\,dy$$

(+ no b.c.s due to compact support)

$$= \int_\Omega \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 dx\,dy$$

$$\Rightarrow \|\Delta u\|_{0,\Omega} \geq |u|_{2,\Omega} \quad \forall u \in C_0^\infty(\Omega)$$

2 × mixed derivatives one × mixed derivative

Can show for $u \in H_0^2(\Omega)$. — $C_0^\infty(\Omega)$ dense in $H_0^2(\Omega)$.