Examples of finite elements

n-simplex

Study finite elements for various values of k. For given k:

- determine principal lattice $L_k(T)$
- express basis functions of the finite element using barycentric coordinates
- draw basis functions of X_h in 2D
- verify $X_h \subset C(\bar{\Omega})$ in 2D
- characterise set of Σ_h of global degrees of freedom

k=1:

- $L_1(T) = \{x = \sum_{j=1}^{n+1} \lambda_j a_j : \lambda_1, \dots, \lambda_{n+1} \in \{0,1\}, \sum_{j=1}^{n+1} \lambda_j = 1\} = \{a_i\}_{i=1}^{n+1}$

To satisfy the condition $\sum_{j=1}^{n+1} \lambda_j = 1$, then one λ_j must be one, and the rest zero $\Rightarrow n+1$ complex combination

- Basis functions defined as 1 at one point of principal lattice, and zero at rest.

$$L_k(T) = \{z_i\}_{i=1}^{M_{n,k}} \quad \text{basis functions satisfying}$$
$$p_i(z_j) = \delta_{ij} \quad i,j = 1, \dots, M_{n,k}$$

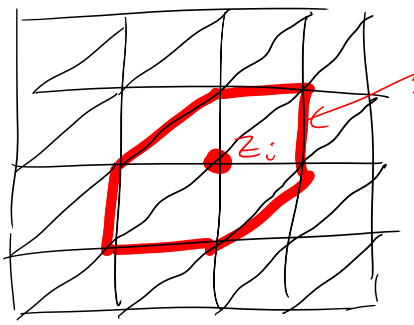
Express basis by barycentric coordinates

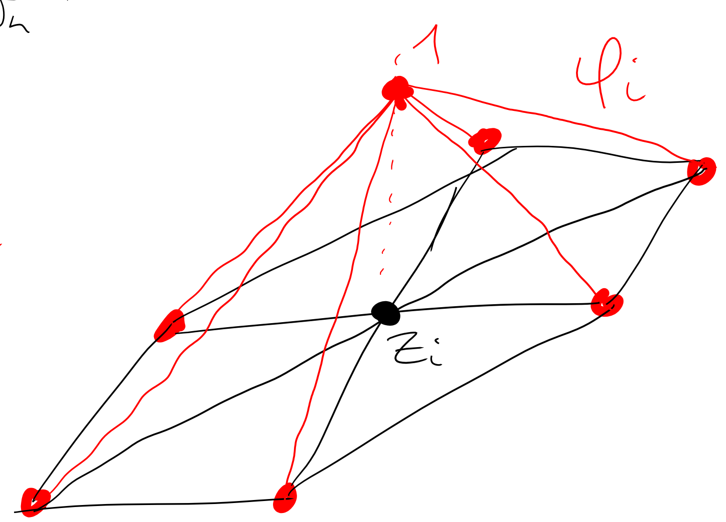
$$\lambda_i \in P_1(T),\ i = 1, \dots, n+1 \quad \lambda_i(a_j) = \delta_{ij} \quad i,j = 1, \dots, n+1$$

Basis functions on finite elements are barycentric coordinate functions:

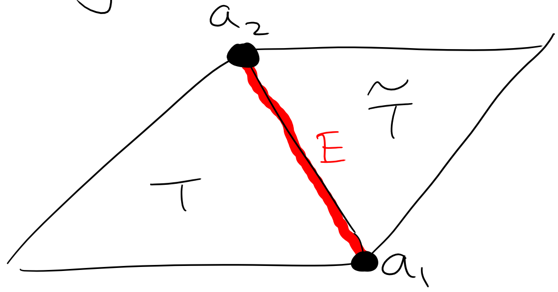
- Basis functions of X_h: if $\bigcup_{T \in \mathcal{T}_h} L_k(T) = \{z_i\}_{i=1}^{N_h}$ then $\varphi_i(z_j) = \delta_{ij}$ $i,j = 1, \dots, N_h$

$$\Rightarrow \operatorname{supp} \varphi_i = \bigcup_{T \in \tilde{\mathcal{T}}_h} T$$





- Continuity of functions in X_h (2D)



Show continuity across an edge shared by elements T & $\tilde{T}$ $\Rightarrow$ true for all edges $\Rightarrow$ continuity of X_h

$$v_h|_T \in P_1(T), \quad v_h|_{\tilde{T}} \in P_1(\tilde{T})$$

$$v_h|_T(a_i) = v_h|_{\tilde{T}}(a_i), \quad i = 1,2$$

$$\Rightarrow p := (v_h|_T)|_E - (v_h|_{\tilde{T}})|_E \in P_1(E)$$

$$p(a_1) = p(a_2) = 0$$

$\rightarrow$ 1D linear function which is zero at two points

$$\Rightarrow p \equiv 0 \text{ on } E$$

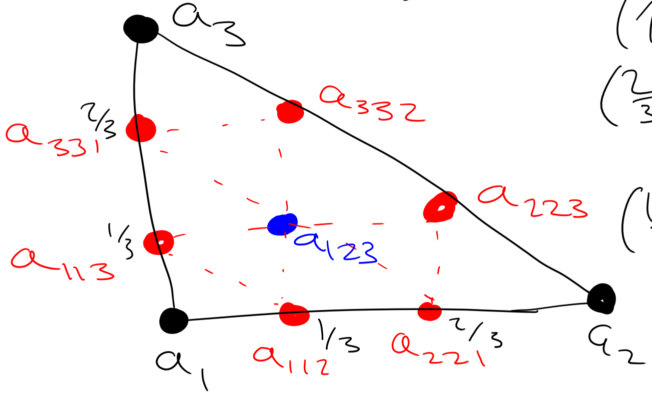
$\Rightarrow v_h$ is continuous across E

- Set of global DoFs – values at vertices of the triangulation $\rightarrow$ as any function from X_h written as sum of basis functions (values at vertices)

$$\Sigma_h = \{\text{values at vertices of } \mathcal{T}_h\}$$

$\underline{k=3}$

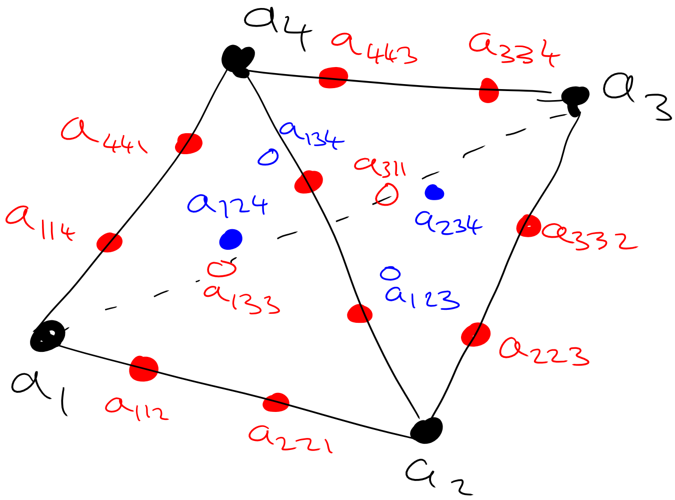
- $L_3(T) = \{ x = \sum_{j=1}^{n+1} \lambda_j a_j : \lambda_1, \dots, \lambda_{n+1} \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}, \sum_{j=1}^{n+1} \lambda_j = 1 \}$



$(1,0,\dots,0)$ vertices: a_i, $i = 1, \dots, n+1$

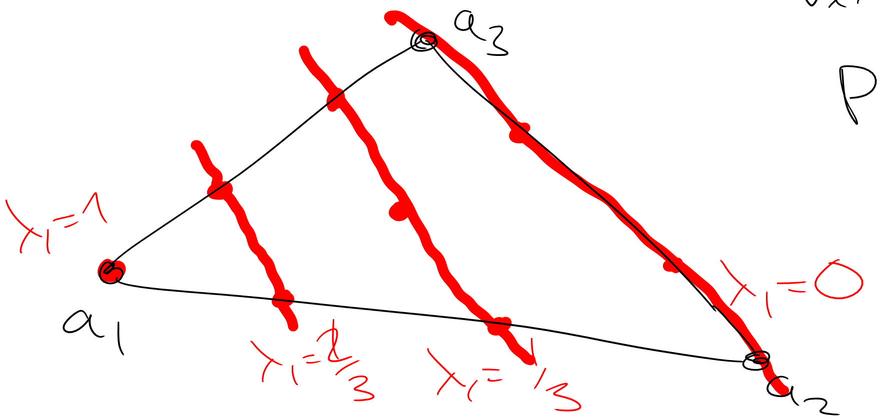
$(\frac{2}{3}, \frac{1}{3}, \dots, 0)$ edges: $a_{iij} = \dfrac{2a_i + a_j}{3}$ for $i,j = 1, \dots, n+1$, $i \neq j$

$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots, 0)$ interior/face: $a_{ijk} = \dfrac{a_i + a_j + a_k}{3}$ for $i,j,k = 1, \dots, n+1$, $i<j<k$



$$L_3(T) = \{a_i\}_{i=1}^{n+1} \cup \{a_{iij}\}_{i \neq j} \cup \{a_{ijk}\}_{i<j<k}$$

- Construct basis that are $\underline{\text{one}}$ at a point in the lattice and zero at all others:



Vertex basis functions:

$$p_i = \frac{9}{2} \lambda_i (\lambda_i - \tfrac{1}{3})(\lambda_i - \tfrac{2}{3}) = \begin{cases} 0 & \text{along } \lambda_i = 0, \frac{1}{3}, \frac{2}{3} \\ 1 & \text{at } a_i \end{cases}$$

Edge basis functions:

$$p_{iij} = \frac{27}{2} \lambda_i \lambda_j (\lambda_i - \tfrac{1}{3}) = \frac{9}{2} \lambda_i \lambda_j (3\lambda_i - 1)$$

0 along edge containing a_j — 0 along edge containing a_i — 0 at barycentre & a_{jji}

Interior/face basis functions: $p_{ijk} = 27 \overbrace{\lambda_i \lambda_j \lambda_k}^{\text{vanishes on edges}}$

Derived in 2D - show true in any dimension:

$$x = \sum_{j=1}^{n+1} \alpha_j a_j, \quad \sum_{j=1}^{n+1} \alpha_j = 1 \qquad \lambda_i(x) = \sum_{k=1}^{n} b_k x_k + b_{n+1}$$
$$= b \cdot x + b_{n+1}$$

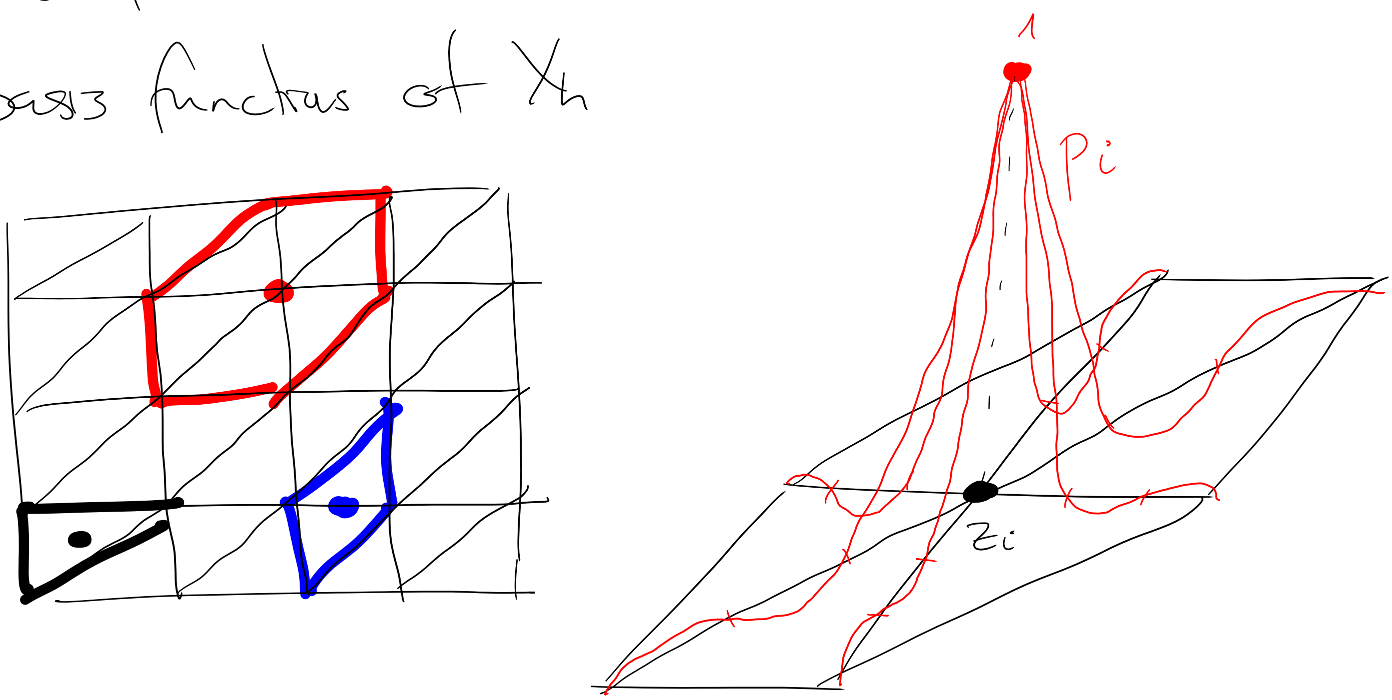
$$\Rightarrow \lambda_i(x) = \sum_{j=1}^{n+1} \alpha_j \left(b \cdot a_j + b_{n+1} \right)$$
$$= \sum_{j=1}^{n+1} \alpha_j \lambda_i(a_j)$$

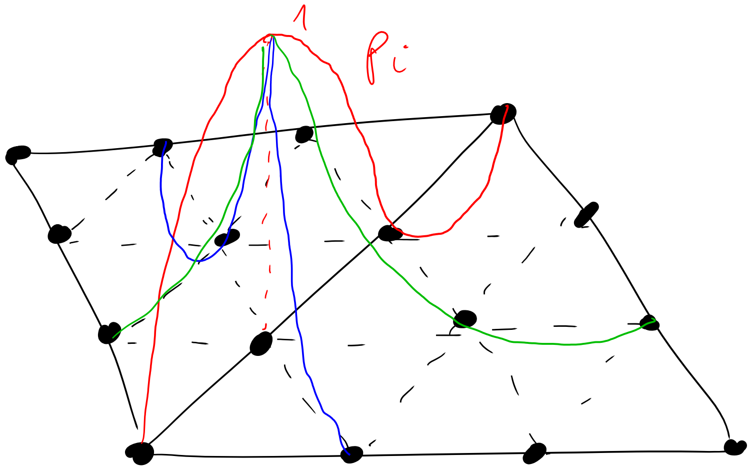
$$\Rightarrow \lambda_\ell(a_{iij}) = \frac{2}{3} \lambda_\ell(a_i) + \frac{1}{3} \lambda_\ell(a_j)$$
$$= \frac{2}{3} \delta_{\ell i} + \frac{1}{3} \delta_{\ell j}$$
$$= \begin{cases} 2/3 & \ell = i \\ 1/3 & \ell = j \\ 0 & \text{otherwise} \end{cases}$$

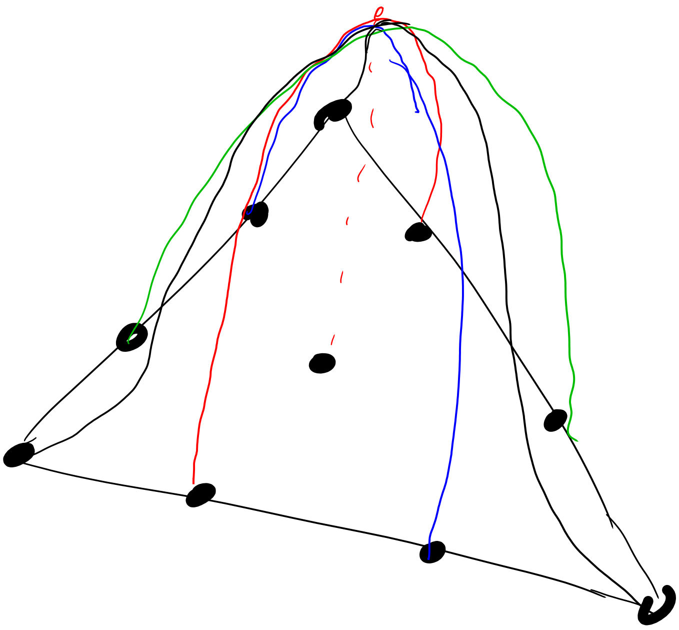
Similarly $\lambda_\ell(a_{ijk}) = \begin{cases} 1/3 & \ell \in \{i, j, k\} \\ 0 & \text{otherwise} \end{cases}$

Insert into p_i, p_{iij} & p_{ijk} for any point validates basis functions.

- Basis functions of X_h

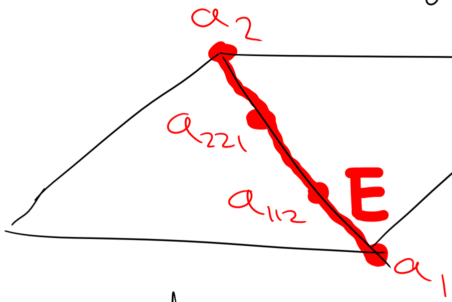






- Continuity (2D):

Again consider an edge E shared by $T, \tilde{T} \in \mathcal{T}_h$



We have that

$$v_h|_T(a_i) = v_h|_{\tilde{T}}(a_i) \qquad i = 1,2$$

$$v_h|_T(a_{iij}) = v_h|_{\tilde{T}}(a_{iij}) \qquad i,j = 1,2,\ i \neq j$$

Again, define

$$p := (v_h|_T)|_E - (v_h|_{\tilde{T}})|_E \in \mathbb{P}_3(E)$$

$$p(a_1)=p(a_2)=p(a_{112})=p(a_{221})=0$$

p is a cubic function with four zeros

$\Rightarrow p\equiv 0 \Rightarrow$ continuity over E

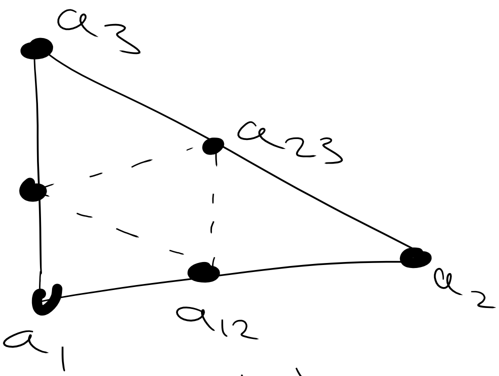
- $\Sigma_h = \{$values at vertices$\} \cup \{$values at barycentre of elements$\}$ $\cup\{$values at points $1/3$ & $2/3$ along each edge$\}$.

$\underline{k=2}$

$$L_2(T)=\left\{x=\sum_{j=1}^{n+1}\lambda_j a_j,\ \lambda_j\in\{0,\tfrac12,1\},\ j=1,\dots,n+1,\ \sum_{j=1}^{n+1}\lambda_j\right\}$$

$$=\{a_i\}_{i=1}^{n+1}\cup\{a_{ij}\}_{i<j}$$

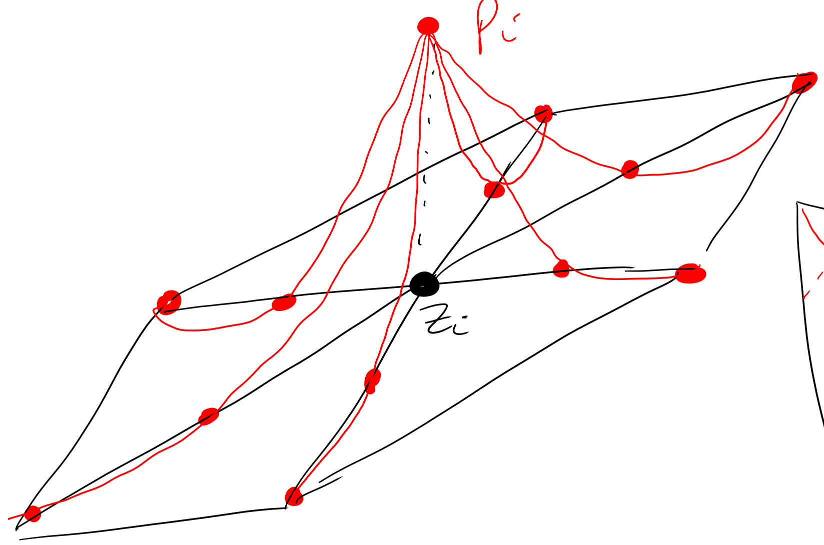
where $a_{ij}=\frac{1}{2}(a_i+a_j)$ $\quad i<j$

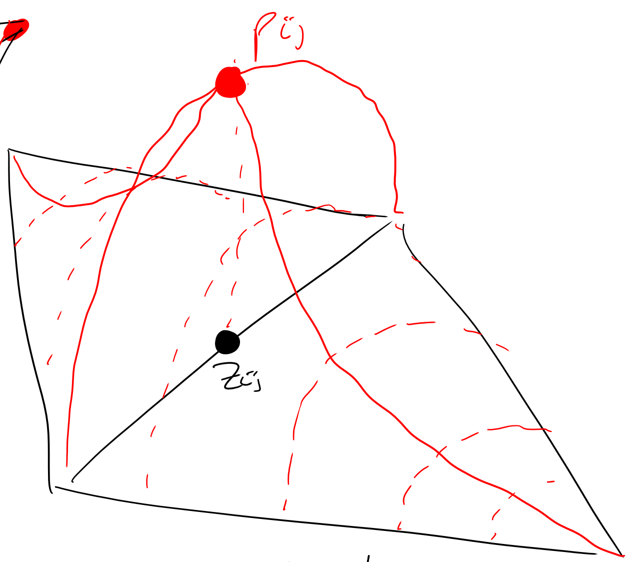


Basis functions: $p_i=\lambda_i(2\lambda_i-1)$, $p_{ij}=4\lambda_i\lambda_j$

$\Rightarrow p_i(a_k)=\delta_{ik}$, $p_i(a_{kl})=0$

& $p_{il}(a_k)=0$, $p_{ij}(a_{kl})=\begin{cases}1 & \text{if } i=k, j=l\\ 0 & \text{otherwise}\end{cases}$





$\Sigma_h=\{$values at vertices$\}\cup\{$values at midpoints of edges$\}$

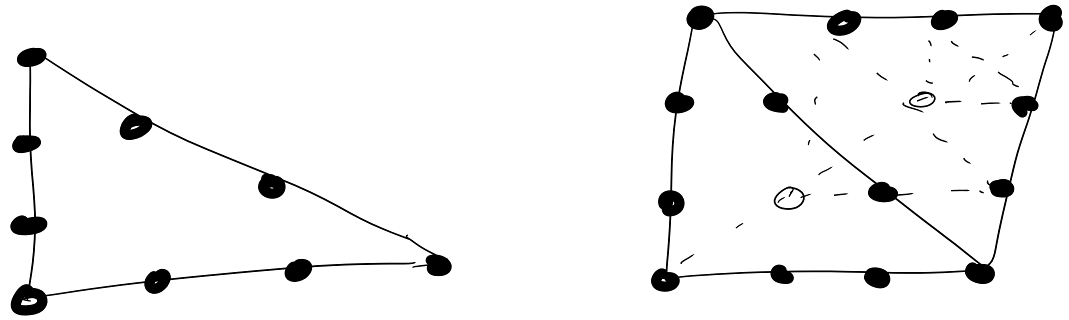
Reduced n-simplex

Have n-simplex T with vertices $\{a_i\}_{i=1}^{n+1}$, interior/face lattice points $a_{ijk} = \frac{1}{3}(a_i + a_j + a_k)$, $i<j<k$ and edge lattice points $a_{iij} = \frac{1}{3}(2a_i + a_j)$, $i \neq j$.

Theorem For any triple $\{i,j,k\}$, $1 < i < j < k < n+1$ let
$$\phi_{ijk}(p) = 12p(a_{ijk}) + 2\sum_{l \in \{i,j,k\}} p(a_l) - 3\sum_{\substack{l,m \in \{i,j,k\} \\ l \neq m}} p(a_{llm})$$

Then, any polynomial in the space
$$P_3'(T) = \{p \in P_3(T) : \phi_{ijk}(p) = 0, 1 \leq i < j < k \leq n+1\}$$
is uniquely determined by its values at the points $\{a_i\}_{i=1}^{n+1} \cup \{a_{ijk}\}_{i \neq j}$. Moreover $P_2(T) \subset P_3'(T)$

Remark The space P_3' is the reduced cubic n-simplex, see Fig 6.4:



Proof • Space uniquely determined by points (see Homework)

• Show $P_2(T) \subset P_3'(T)$. Let $p \in P_2(T)$ and let $A = \nabla^2 p$ = constant. Consider any $I = \{i,j,k\}$
(hessian)
with $0 < j < k$. By Taylor's formula for any $x \in T$ at a_{ijk}
$$p(x) = p(a_{ijk}) + \nabla p(a_{ijk}) \cdot (x - a_{ijk}) + \frac{1}{2}(x - a_{ijk}) \cdot A(x - a_{ijk})$$

$$\sum_{\ell\in I} p(a_\ell) = \underline{3p(a_{ijk})} + \nabla p(a_{ijk}) \cdot \underbrace{\Big(\underbrace{\sum_{\ell\in I} a_\ell}_{=a_i+a_j+a_k} - 3\underbrace{a_{ijk}}_{=\frac{1}{3}(a_i+a_j+a_k)}\Big)}_{=0}$$

$$+ \underline{\sum_{\ell\in I} \frac{1}{2}(a_\ell - a_{ijk})\cdot A(a_\ell - a_{ijk})}$$

$$\sum_{\substack{\ell,m\in I\\ \ell\neq m}} p(a_{\ell\ell m}) = \underline{6p(a_{ijk})} + \nabla p(a_{ijk}) \cdot \underbrace{\Big(\sum_{\substack{\ell,m\in I\\ \ell\neq m}} a_{\ell\ell m} - 6a_{ijk}\Big)}_{=0}$$

$$\underbrace{+ \sum_{\substack{\ell,m\in I\\ \ell\neq m}} \frac{1}{2}(a_{\ell\ell m} - a_{ijk})\cdot A(a_{\ell\ell m} - a_{ijk})}$$

$$\Rightarrow = \underline{\frac{1}{3}\sum_{\ell\in I}(a_\ell - a_{ijk})\cdot A(a_\ell - a_{ijk})}$$

$$\boxed{\sum_{\substack{\ell,m\in I\\ \ell\neq m}} a_{\ell\ell m} = \sum_{\substack{\ell,m\in I\\ \ell\neq m}} \frac{2}{3}a_\ell + \sum_{\substack{\ell,m\in I\\ \ell\neq m}} \frac{1}{3}a_m = \frac{4}{3}\sum_{\ell\in I} a_\ell + \frac{2}{3}\sum_{m\in I} a_m = 2\sum_{\ell\in I} a_\ell = 6a_{ijk}}$$

Use identity:

$$a_{\ell\ell m} - a_{ijk} = \frac{2}{3}a_\ell + \frac{1}{3}a_m - a_{ijk} = \frac{1}{3}\big[2(a_\ell - a_{ijk}) + (a_m - a_{ijk})\big]$$

$$\Rightarrow \sum_{\substack{\ell,m\in I\\ \ell\neq m}} (a_{\ell\ell m} - a_{ijk})\cdot A(a_{\ell\ell m} - a_{ijk})$$

$$= \frac{1}{9}\sum_{\substack{\ell,m\in I\\ \ell\neq m}} \Big[\overbrace{4(a_\ell - a_{ijk})\cdot A(a_\ell - a_{ijk})}^{\text{ind. of } m} + 4(a_\ell - a_{ijk})\cdot A(a_m - a_{ijk}) + \underbrace{(a_m - a_{ijk})\cdot A(a_m - a_{ijk})}_{\text{ind. of } \ell}\Big]$$

$$= \frac{10}{9}\sum_{\ell\in I}(a_\ell - a_{ijk})\cdot A(a_\ell - a_{ijk}) + \frac{4}{9}\sum_{\ell\in I}(a_\ell - a_{ijk})\cdot A\Big(\underbrace{\sum_{\substack{m\in I\\ m\neq \ell}} a_m - 2a_{ijk}}_{\substack{=(\sum_{m\in I} a_m) - a_\ell - 2a_{ijk}\\ = a_{ijk} - a_\ell}}\Big)$$

$$= \frac{6}{9}\sum_{\ell\in I}(a_\ell - a_{ijk})\cdot A(a_\ell - a_{ijk})$$

Now,

$$\phi_{ijk}(p) = \underbrace{12p(a_{ijk})} + \underbrace{\underbrace{2\sum_{l \in I} p(a_l)}_{6p(a_{ijk})}}_{2 \cdot \frac{1}{2}\, \bullet} - \underbrace{\underbrace{3\sum_{\substack{l,m \in I \\ l = m}} p(a_{lm})}_{18p(a_{ijk})}}_{3 \cdot \frac{1}{3}\, \bullet} = 0$$