# FE for fourth order

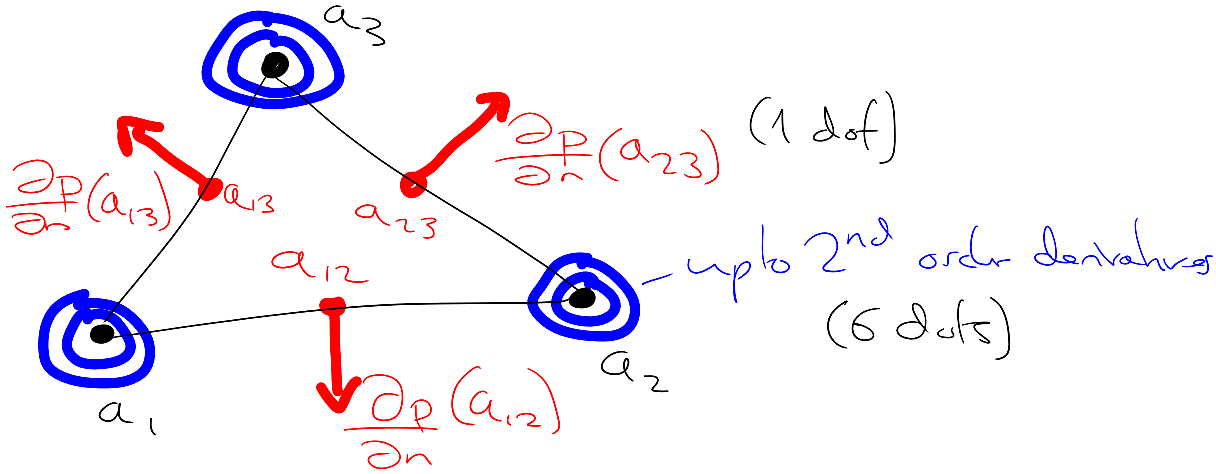
Need $X_h \subset H^2(\Omega) \Rightarrow$ Construct $X_h \subset C^1(\overline{\Omega})$ consisting of piecewise $H^2$ functions. Consider 2D only.

## Argyris Triangle (Fig 9.1)

**Lemma** Let $T$ be a triangle with vertices $a_1, a_2, a_3$ and $a_{ij} = \frac{1}{2}(a_i + a_j)$, $1 \le i < j \le 3$. Then, any polynomial $p \in P_5$ uniquely determined by following DoF:

$$\Sigma_T = \{D^\alpha p(a_i), |\alpha| \le 2, i = 1,..,3; \frac{\partial p}{\partial n}(a_{ij}), 1 \le i < j \le 3\}$$

where $n$ is the normal to $\partial T$.




**Proof**

$$\text{card } \Sigma_T = 21 = \dim P_5 = \binom{5+2}{2} = \binom{7}{2}$$

If all dofs vanish for some $p \in P_5$ then $p \equiv 0$ (unisolvence):

Let $p \in P_5$ such that

$$D^\alpha p(a_i) = 0, |\alpha| \le 2, i = 1, 2, 3, \frac{\partial p}{\partial n}(a_{ij}) = 0, 1 \le i < j \le 3$$

① Show $p=0$ on $\partial T$ & $\nabla p = 0$ on $\partial T$

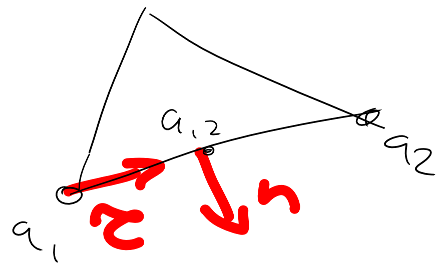
Let $q := p|_{[a_1,a_2]}$ ($a_1, a_2$ are arbitrary). Then, $q \in P_5$

& $q(a_1) = q'(a_1) = q''(a_1) = q(a_2) = q'(a_2) = q''(a_2) = 0$

$\Rightarrow q = 0$

$\Rightarrow p = 0$ on $\partial T$

$$\left( \begin{array}{l} q'(a_1) = \frac{\partial p}{\partial \tau}(a_1) \\ q''(a_2) = \frac{\partial^2 p}{\partial \tau^2}(a_2) \end{array} \right.$$




Set $r := \frac{\partial p}{\partial n}\Big|_{[a_1,a_2]}$. Then, $r \in P_4$

$r(a_1) = r'(a_1) = r(a_{12}) = r'(a_2) = r(a_2) = 0$

where $r(a_1) = \frac{\partial p}{\partial n}(a_1)$, $r'(a_1) = \frac{\partial^2 p}{\partial \tau \partial n}(a_1)$, $r(a_{12}) = \frac{\partial p}{\partial n}(a_{12})$

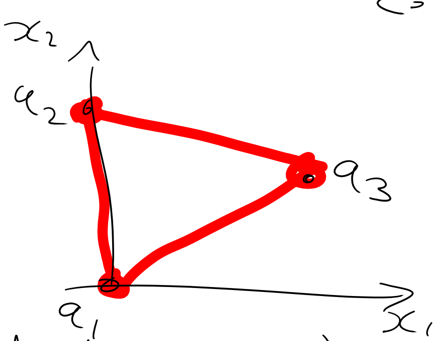
$$\left.\begin{array}{l} \Rightarrow r \equiv 0 \quad \Rightarrow \frac{\partial p}{\partial n} = 0 \text{ on } \partial T \\ \text{\& as } p \equiv 0 \text{ on } \partial T \text{ then } \frac{\partial p}{\partial \tau} = 0 \text{ on } \partial T \end{array}\right\} \Rightarrow \nabla p = 0 \text{ on } \partial T.$$

② Show $p = 0$ in $T$

w.l.o.g assume that edge $[a_1, a_2]$ lies on axis $x_2$.

p can be expressed as

$$p(x_1, x_2) = \sum_{i=0}^{5} x_1^i p_i(x_2)$$




where $p_i$, $i = 0, \dots, 5$ are polynomials of degree $(5-i)$ in $x_2$

$$\left.\begin{array}{l} \text{Then, } 0 = p(0, x_2) = p_0(x_2) \\ 0 = \frac{\partial p}{\partial x_1}(0, x_2) = p_1(x_2) \end{array}\right\} \Rightarrow p(x_1, x_2) = x_1^2 \sum_{i=2}^{5} x_1^{i-2} p_i(x_2)$$

Since $\lambda_3(x) = \alpha x_1$, we see that $p = \lambda_3^2 \tilde{p}$, $\tilde{p} \in P_3$

Easy to see $\tilde{p} = 0$, $\nabla \tilde{p} = 0$ on $[a_1, a_3]$ and $[a_2, a_3]$

Similar to before can show $\tilde{p} = \lambda_2^2 \bar{p}$, $\bar{p} \in P_1$

Can show $\bar{p} = 0$ & $\nabla \bar{p} = 0$ on $[a_2, a_3] \Rightarrow \bar{p} = 0$ in $T$

$\Rightarrow p = 0$ in $T$ $\square$

Consider two Argyris triangles $(T, P_T, \Sigma_T)$, $(\hat{T}, \hat{P}_T, \hat{\Sigma}_T)$

$T$-vertices $a_1, a_2, a_3$ -- $\hat{T}$-vertices $\hat{a}_1, \hat{a}_2, \hat{a}_3$

$P_T = P_5(T)$ $\qquad \hat{P}_T = P_5(\hat{T})$

$$\Sigma_T = \{ p(a_i) : i = 1,2,3;\ \nabla p(a_i) \cdot (a_j - a_i)\ i,j = 1,2,3, i \neq j;$$
$$(a_k - a_i) \cdot \nabla^2 p(a_i)(a_j - a_i)\ i,j,k = 1,2,3, i \neq j, i \neq k, j \leq k;$$
$$\nabla p(a_{ij}) \cdot (a_j - a_i)^\perp, i = 1,2,3, i < j \}$$

where

- $\nabla^2 p$ - Hessian of $p$
- $a_{ij} = \frac{1}{2}(a_i + a_j)$
- $u^\perp = (u_2, -u_1)$ $\forall u \in \mathbb{R}^2$

when are they affine equivalent?

- $\exists$ linear affine mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $F(\hat{T}) = T$ and $a_i = F(\hat{a}_i)$ $i = 1,2,3$.
- Clearly $\{ \hat{p} \circ F^{-1} : \hat{p} \in \hat{P}_T \} = P_T$ just need to investigate $\Sigma_T = \{ \phi : \phi(p) = \hat{\phi}(p \circ F), \hat{\phi} \in \hat{\Sigma}_T \}$
- $\hat{\phi}(\hat{p}) = \hat{p}(\hat{a}_i)$ then $\hat{\phi}(p \circ F) = (p \circ F)(\hat{a}_i) = p(F(\hat{a}_i))$
$= p(a_i) = \phi(p)$

$-\ \hat\phi(\hat p) = \hat\nabla \hat p(\hat a_i)\cdot(\hat a_j - \hat a_i)$; then

$$\hat\phi(p\circ F) = \hat\nabla(p\circ F)(\hat a_i)\cdot(\hat a_j - \hat a_i)$$

$$= \sum_{k,l=1}^{n} \frac{\partial p}{\partial x_l}(\underbrace{F(\hat a_i)}_{=a_i})\, \frac{\partial F_l}{\partial \hat x_k}(\hat a_i)(\hat a_j - \hat a_i)_k$$

Since $F(\hat x) = \mathbb{B}\hat x + b \Rightarrow \frac{\partial F_l}{\partial x_k} = B_{lk}$ and, hence,

$$\hat\phi(p\circ F) = \sum_{l=1}^{n} \frac{\partial p}{\partial x_l}(a_i)(\mathbb{B}\hat a_j - \mathbb{B}\hat a_i)_l$$

$$= \sum_{l=1}^{n} \frac{\partial p}{\partial x_l}(a_i)(F(\hat a_j) - F(\hat a_i))_l$$

$$= \nabla p(a_i)\cdot(a_j - a_i)$$

$-$ If $\hat\phi(\hat p) = (\hat a_k - \hat a_i)\cdot\hat\nabla^2 \hat p(a_i)(a_j - a_i)$; then,

$$\hat\phi(p\circ F) = \sum_{l,m=1}^{n} (\hat a_k - \hat a_i)_l \cdot \frac{\partial^2 (p\circ F)}{\partial \hat x_k \partial \hat x_m}(\hat a_i)(\hat a_j - \hat a_i)_m$$

$$= \sum_{l,m=1}^{n}\sum_{r,s=1}^{n} (\hat a_k - \hat a_i)_l \frac{\partial^2 p}{\partial x_s \partial x_r}(F(\hat a_i))\, B_{sl} B_{rm} (\hat a_j - \hat a_i)_m$$

$$= \sum_{r,s=1}^{n} (\mathbb{B}\hat a_k - \mathbb{B}\hat a_i)_s \frac{\partial^2 p}{\partial x_s \partial x_r}(a_i)(\mathbb{B}\hat a_j - \mathbb{B}\hat a_i)_r$$

$$= (a_k - a_i)\cdot\nabla^2 p(a_i)(a_j - a_i)$$

$-$ If $\hat\phi(\hat p) = \hat\nabla \hat p(\hat a_{ij})\cdot(\hat a_j - \hat a_i)^\perp$ Similarly to previous:

$$\hat\phi(p\circ F) = \nabla p(a_{ij})\cdot\mathbb{B}(\hat a_j - \hat a_i)^\perp$$

Therefore, the two triangles are affine equivalent when $\mathbb{B}(\hat a_j - \hat a_i)^\perp = (a_j - a_i)^\perp$ $\quad i = 1,2,3$

Since $a_j - a_i = \mathbb{B}(\hat{a}_j - \hat{a}_i)$ only need to consider when $\mathbb{B}e_i^\perp = (\mathbb{B}e_i)^\perp$ $i=1,2$ with $e_1=(1,0), e_2=(0,1)$

True if and only if $b_{11}=b_{22}$ & $b_{12}=-b_{21}$

So only admits:

- uniform scaling $(b_{12}=-b_{21}=0,\ b_{11}=b_{22}=s, s\in\mathbb{R})$
- rotation $(b_{11}=b_{22}=\cos\theta,\ b_{12}=-b_{21}=\sin\theta,\ \theta\in[0,2\pi])$
- translation $(\mathbb{B}=\mathbb{I})$.

## Bogner-Fox-Schmidt Rectangle (Fig 9.4)

$T$ rectangle with vertices $a_1, \dots, a_4$. Then any polynomial $p \in Q_3$ uniquely determined by following DoFs

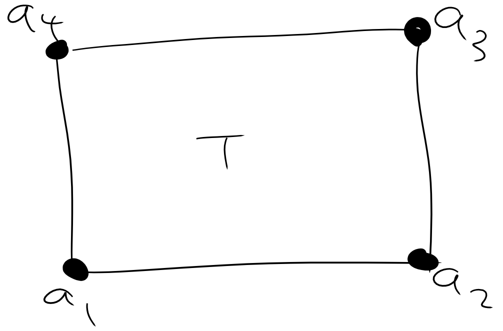
$$\Sigma = \left\{ p(a_i), \frac{\partial p}{\partial x_1}(a_i), \frac{\partial p}{\partial x_2}(a_i), \frac{\partial^2 p}{\partial x_1 \partial x_2}(a_i) : i=1,\dots,4 \right\}$$

Corresponding space satisfies $X_h \subset C^1(\Omega) \cap H^2(\Omega)$.

Proof $\dim Q_3 = 4^2 = 16 = \operatorname{card} \Sigma$. Let $p \in Q_3$ such that

$$p(a_i) = \frac{\partial p}{\partial x_1}(a_i) = \frac{\partial p}{\partial x_2}(a_i) = \frac{\partial^2 p}{\partial x^2}(a_i) = 0 \quad i=1,\dots,4$$

want to show $p \equiv 0 \implies$ unisolvence




Let $q$ be restriction of $p$ to edge $[a_1, a_2]$. Then $q \in P_3$ and $q(a_1) = q(a_2) = q'(a_1) = q'(a_2) = 0$

$\implies q \equiv 0$ on $[a_1, a_2]$. $\implies p \equiv 0$ on $\partial T$

Let $r$ be restriction of $\frac{\partial p}{\partial n}$ to $[a_1, a_2]$, where $n$ normal to $[a_1, a_2]$.

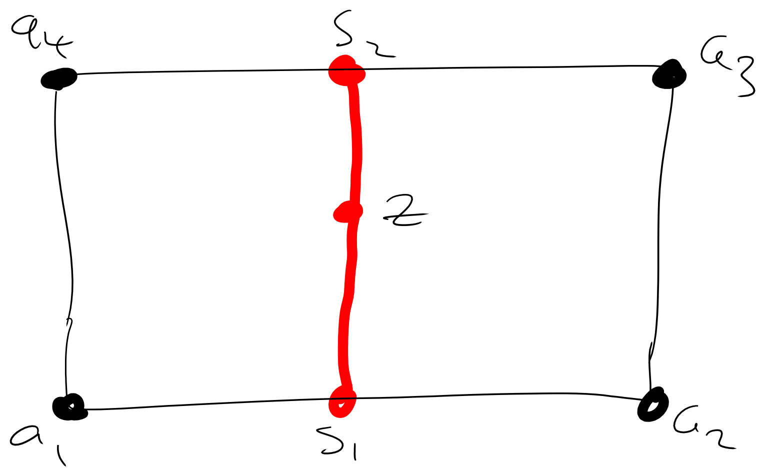
Then $r \in P_3$ & $r(a_1) = r(a_2) = r'(a_1) = r'(a_2) = 0 \Rightarrow r \equiv 0$ on $\partial T$

$\Rightarrow \frac{\partial p}{\partial n} = 0$ on $\partial T$

$p = 0$ on $\partial T \Rightarrow \frac{\partial p}{\partial \tau} = 0$ on $\partial T$ $\Bigg\} \Rightarrow \nabla p = 0$ on $\partial T$

(where $\tau$ is vector along $[a_1, a_2]$)

Let $z \in \operatorname{int} T$ and $S$ be line crossing $z$ parallel to $[a_2, a_3]$



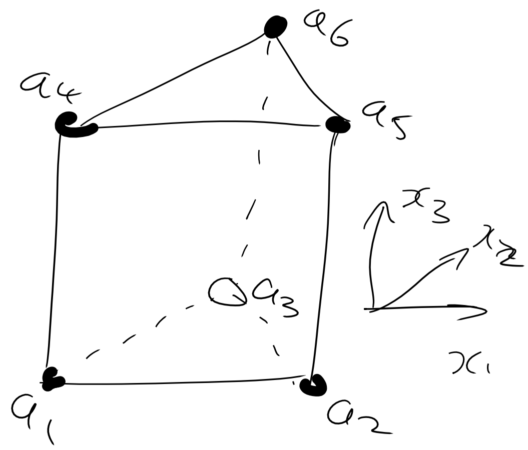

Then $p|_S(s_1) = p|_S(s_2) = p'|_S(s_1) = p'|_S(s_2) = 0$

Since, $p|_S \in P_3$, $p = 0$ on $S \Rightarrow p(z) = 0$

As $z$ is arbitrary $\Rightarrow p \equiv 0$ on $T$.

↳ Can show continuity over edges of $v_h \in X_h$ and continuity of $\nabla v_h$

## FE on prism



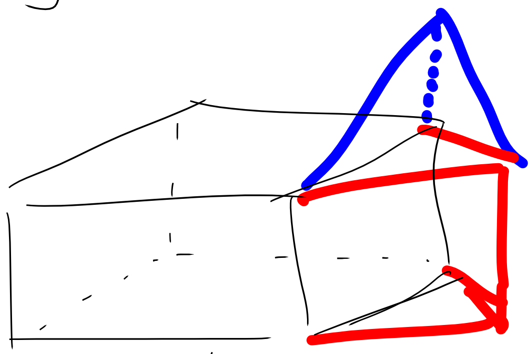

$T$ - prism - pentahedron

$$P_T = \{ p(x_1, x_2, x_3) = \gamma_1 + \gamma_2 x_1 + \gamma_3 x_2 + \gamma_4 x_4 + \gamma_5 x_1 x_3 + \gamma_6 x_2 x_3 : \gamma_1, \ldots \gamma_6 \in \mathbb{R} \}$$

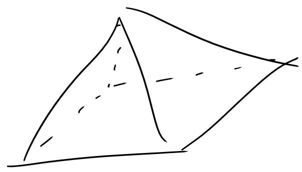
- Cartesian product $\underset{(x_1, x_2)}{P_1(\mathbb{R}^2)} \times \underset{(x_3)}{P_1(\mathbb{R})}$

Then, any function from $P_T$ uniquely determined by values at vertices $a_1, \dots a_6$. Moreover, $\forall p \in P_T$ and faces $F \subset \partial T$, $p|_F$ uniquely determined by values at vertices of $F$ (HW)

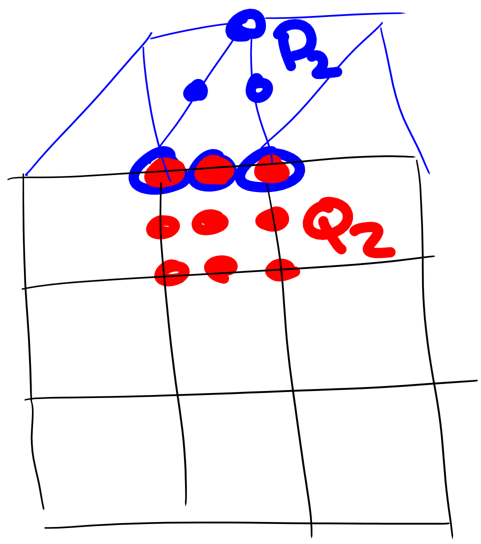
↳ Useful as can construct triangulation combining tetrahedrons, cubes & prisms:



– Another useful element here is pyramid:



In 2D can combine rectangles & triangles directly:




Resulting space satisfies $X_h \subset C(\overline{\Omega})$