Nonconforming Crouzeix – Raviart linear n-simplex

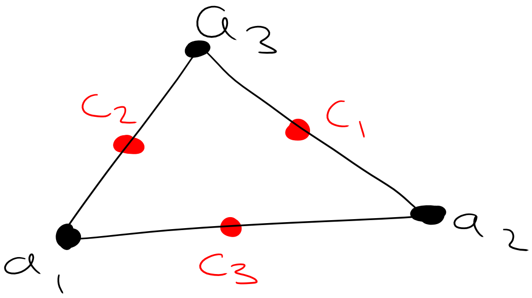
T = n-simplex with simplices $a_1, \dots, a_{n+1}$

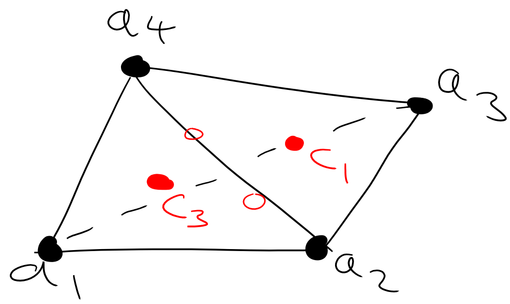
$P_T = P_1(T)$

$\Sigma_T = \{ p(c_i) : i = 1, \dots, n+1 \}$

where c_i is barycentre of face opposite vertex a_i

i.e. $c_1 = \frac{1}{n} \sum_{j=1}^{n+1} a_j$





Then Σ_T is P_T-unisolvent.

Proof Barycentric coordinates satisfy

$$\lambda_k(c_i) = \frac{1}{n} \sum_{j \neq i} \lambda_k(a_j) = \frac{1}{n}\left(\sum_{j=1}^{n+1} \lambda_k(a_j) - \lambda_k(a_i) \right)$$

$$= \frac{1}{n}\left(1 - \lambda_k(a_i)\right)$$

$$\Rightarrow \lambda_k(a_i) = 1 - n\lambda_k(c_i)$$

Set $p_k(x) = 1 - n\lambda_k(x)$. Then,

$$p_k(c_i) = \lambda_k(a_i) = \delta_{ki} \quad i, k = 1, \dots, n+1$$

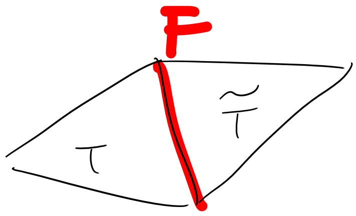
Functions $p_1, \dots, p_{n+1}$ are linearly independent and since $\dim P_1(T) = n+1$ they form a basis of $P_1(T)$ $\square$

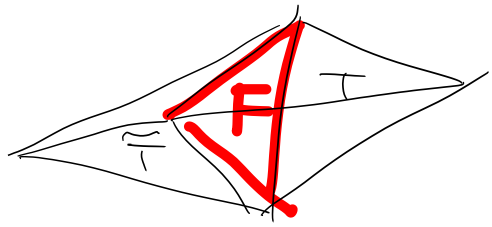
Let $\mathcal{T}_h$ be triangulation of $\Omega \subset \mathbb{R}^n$ consisting of n-simplices. Then, FE space is

$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in P_1(T) \ \forall T \in \mathcal{T}_h, v_h \text{ are continuous at barycentre of faces of } \mathcal{T}_h \}$$

Define jump of v across face F:

$$[\![v]\!]_F = (v_h|_T)|_F - (v_h|_{\tilde{T}})|_F$$





Then, we can formulate X_h as

$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h, \ [\![v]\!]_F(c_F) = 0 \ \forall F \in \mathcal{F}_h^i \}$$

where c_F is barycentre of F & $\mathcal{F}_h^i$ is set of all interior faces of $\mathcal{T}_h$.

Since, for any $p \in \mathbb{P}_1(F)$ $\frac{1}{|F|}\int_F p \, ds = p(c_F)$, then

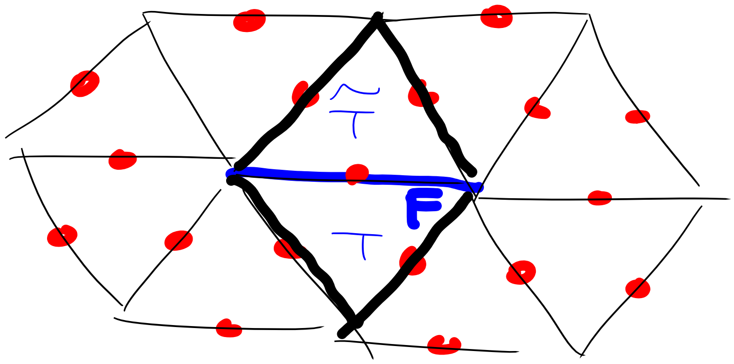
$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in \mathbb{P}_1(T) \ \forall T \in \mathcal{T}_h, \ \int_F [\![v]\!]_F \, ds = 0 \ \forall F \in \mathcal{F}_h^i \}$$

The degrees of freedom of X_h are

$$\Sigma_h = \{ v_h(c_F) : F \in \mathcal{F}_h \} \text{ or } \Sigma_h = \{ \frac{1}{|F|} \int_F v_h \, ds : F \in \mathcal{F}_h \}$$

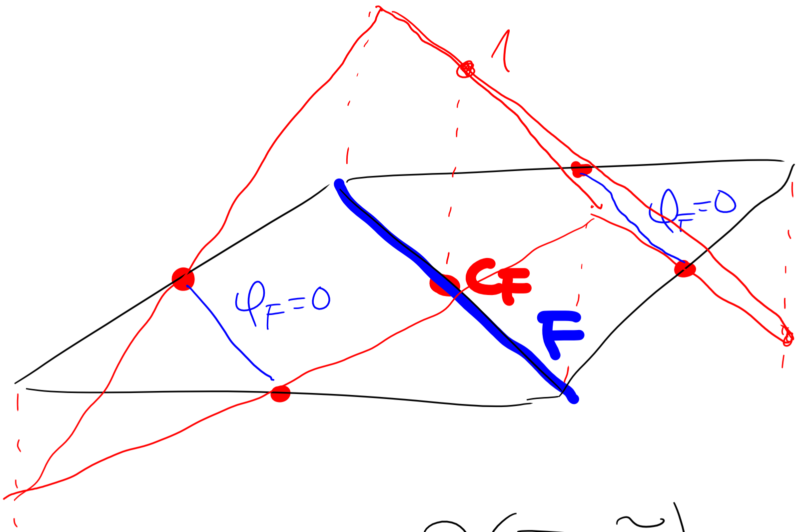
where $\mathcal{F}_h$ is the set of all faces of $\mathcal{T}_h$.

Basis functions of X_h in 2D



$\operatorname{supp} \varphi_F = T \cup \tilde{T}$

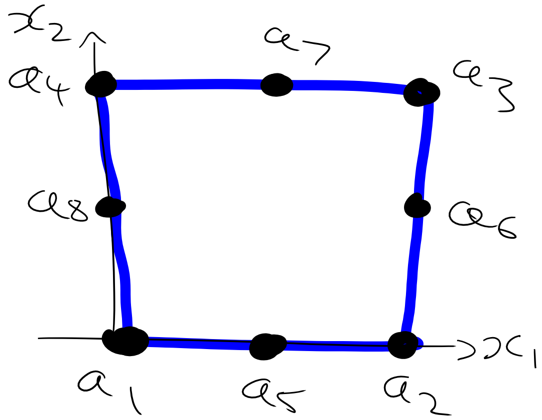
$$\varphi_F(c_{\tilde{F}}) = \begin{cases} 1 & F = \tilde{F} \\ 0 & \text{otherwise} \end{cases}$$



φ_F has jumps across $\partial(T\cup\tilde{T})$

Functions in X_h are <u>discontinuous</u> – jumps across interior faces (only continuous at barycentres of faces).

Example – difficult to construct nonconforming FEs



- $T=[0,1]^2$
  $a_5,\dots,a_8$ midpoints of edges
- $P_T = Q_1(T)$
- $\Sigma_T = \{p(a_i) : i=5,\dots,8\}$

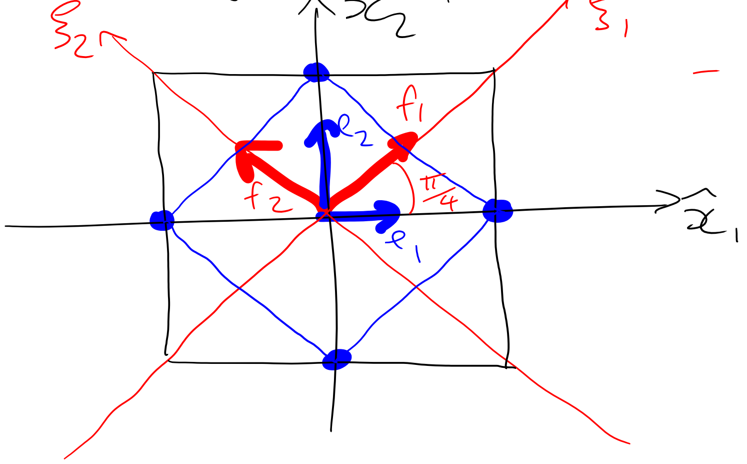
Then, Σ_T is <u>not</u> unisolvent.

<u>Counterexample</u> $p=(x_1-\tfrac{1}{2})(x_2-\tfrac{1}{2})$. Then $p(a_i)=0$, $i=5,\dots,8$.

$p\in P_T$, $p\not\equiv 0$. $\square$

Nonconforming FE on rectangles

Consider $\hat{T}=[-1,1]^2$ and points at midpoints of edge



– Consider alternate coordinate system

Then, the points are the vertices of a "rotated square" (parallel to ξ_1, ξ_2) — use bilinear functions on this alternate coordinate system & transform to original system; then space is P-unisolvent.

e_1, e_2 – unit vectors in $\hat{x}_1, \hat{x}_2$

f_1, f_2 – unit vectors in ξ_1, ξ_2

Coordinates of a point A are $(\hat{x}_1, \hat{x}_2)$ or (ξ_1, ξ_2)

$$A - O = \hat{x}_1 e_1 + \hat{x}_2 e_2 = \xi_1 f_1 + \xi_2 f_2$$

$$\Rightarrow \quad \xi_1 = (A-O)\cdot f_1 = \hat{x}_1 e_1 \cdot f_1 + \hat{x}_2 e_2 \cdot f_2 = \frac{\sqrt{2}}{2}(\hat{x}_1 + \hat{x}_2)$$

$$\xi_2 = (A-O)\cdot f_2 = \hat{x}_1 e_1 \cdot f_2 + \hat{x}_2 e_2 \cdot f_2 = \frac{\sqrt{2}}{2}(-\hat{x}_1 + \hat{x}_2)$$

(transformation between two coordinate systems)

$\Rightarrow$ basis $1, \xi_1, \xi_2, \xi_1\xi_2$ of Q_1 (on ξ_1, ξ_2 coordinate system) expressed in $\hat{x}_1, \hat{x}_2$ coordinate system is

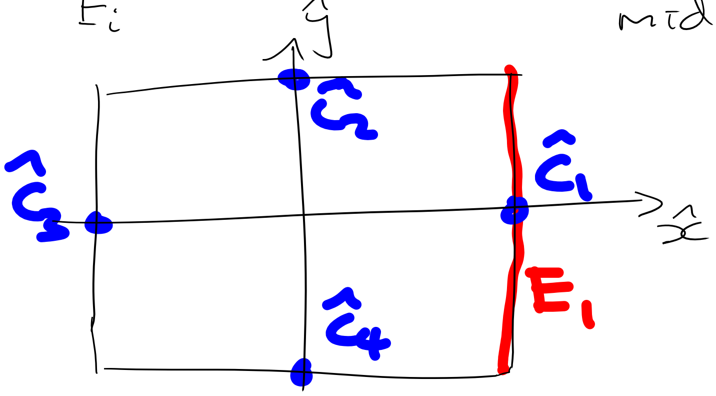
$$1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2 - \hat{x}_2^2$$

This gives <u>Rotated bilinear element</u> – Q_1^{rot}

Let $\hat{T} = [-1,1]^2$, $\hat{P} = \text{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}$

$\hat{\Sigma} = \{\hat{p}(\hat{C}_i) : i = 1, \dots, 4\}$ where $\hat{C}_1 = (1,0)$, $\hat{C}_2 = (0,1)$, $\hat{C}_3 = (-1,0)$, $\hat{C}_4 = (0,-1)$

or $\hat{\Sigma}' = \{\frac{1}{2}\int_{\hat{E}_i} \hat{p}\, ds : i = 1, \dots, 4\}$ where $\hat{E}_i$ is edge of $\hat{T}$ with midpoint $\hat{C}_i$, $i = 1, \dots, 4$.



Unlike Crouzeix-Raviart different basis leads to different values.

- Want to show $\hat{P}_T$-unisolvent, define basis functions, and for triangulation of Ω of equal squares with edge length h and FE space X_h that FE (T, P_T, Σ_T) affine equivalent to $(\hat{T}, \hat{P}, \hat{\Sigma})$ or $(\hat{T}, \hat{P}, \hat{\Sigma}')$.

• Let $J_{\hat{E}_i}(\hat{v}) = \hat{v}(c_i)$ or $J_{\hat{E}_i} = \frac{1}{2}\int_{\hat{E}} \hat{v}\, d\hat{s}$, $i=1,\dots,4$.

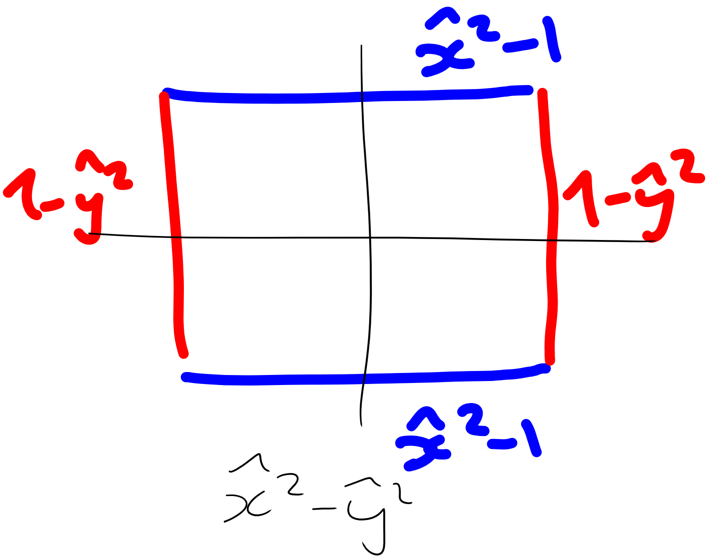
Then $\{J_{\hat{E}_i}, i=1,\dots,4\}$ is $\hat{\Sigma}$ or $\hat{\Sigma}'$. Let $\varphi = \hat{x}^2 - \hat{y}^2$ and $\kappa = 4 J_{\hat{E}_1}(\varphi)$. Then, $\kappa \neq 0$ and

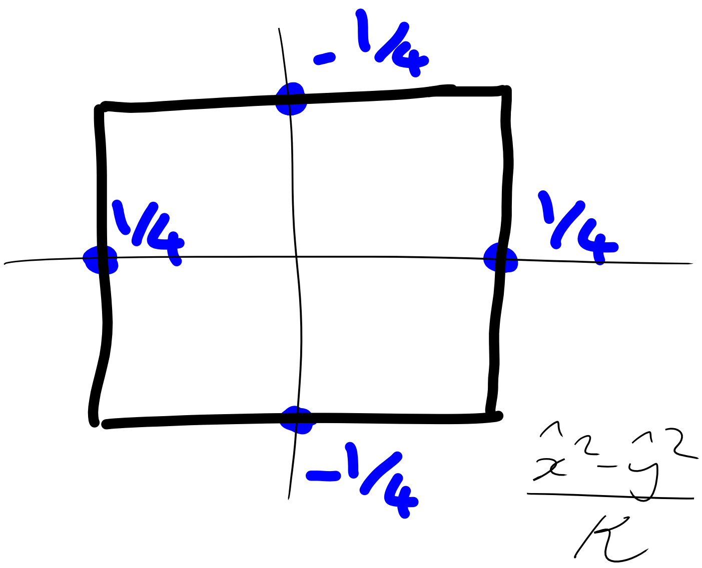
$$\hat{p}_1(\hat{x},\hat{y}) = \frac{1}{4} + \frac{1}{2}\hat{x} + \frac{1}{\kappa}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{p}_2(\hat{x},\hat{y}) = \frac{1}{4} + \frac{1}{2}\hat{y} - \frac{1}{\kappa}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{p}_3(\hat{x},\hat{y}) = \frac{1}{4} - \frac{1}{2}\hat{x} + \frac{1}{\kappa}(\hat{x}^2 - \hat{y}^2)$$
$$\hat{p}_4(\hat{x},\hat{y}) = \frac{1}{4} - \frac{1}{2}\hat{y} - \frac{1}{\kappa}(\hat{x}^2 - \hat{y}^2)$$

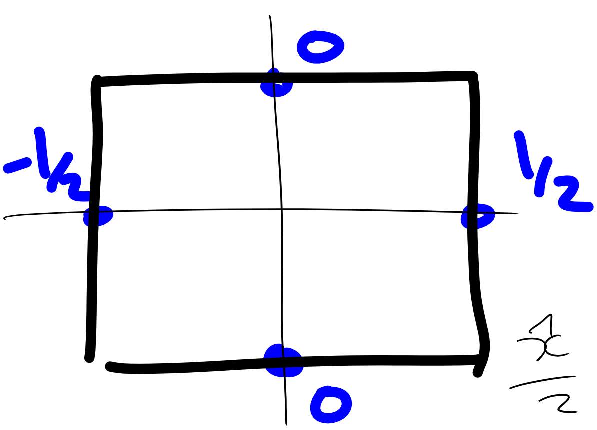
are the basis functions corresponding to the edges

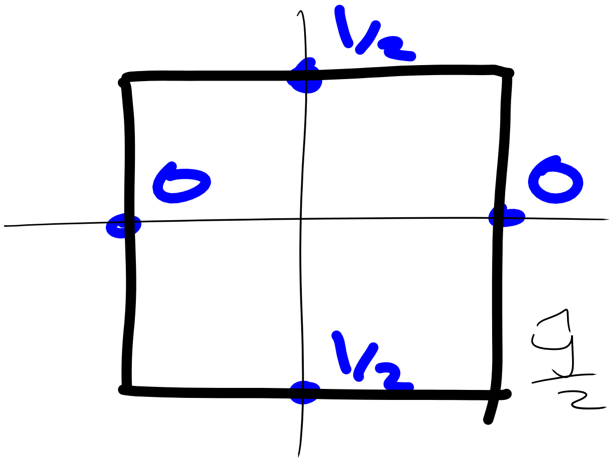
$E_1, \dots, E_4$.

Graphically, prove this basis:





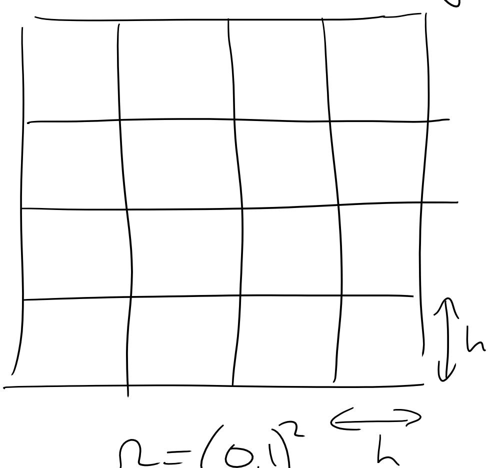




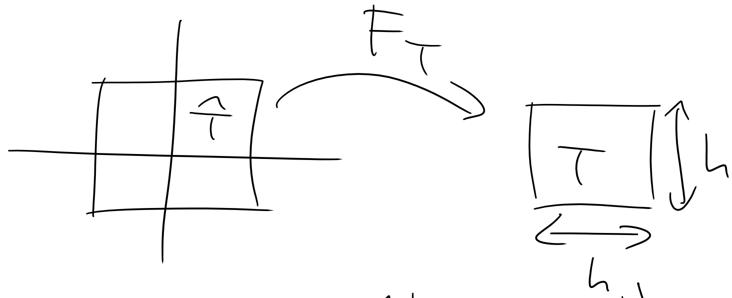
Can check that $\hat{p}_1(\hat{x},\hat{y}) = \begin{cases} 1 & (\hat{x},\hat{y}) = c_1 \\ 0 & \text{otherwise} \end{cases}$

and similarly for $\hat{p}_2, \dots, \hat{p}_4$.

Now consider triangulation $\mathcal{T}_h$



$\Omega = (0,1)^2$



$$F_T(\hat{x},\hat{y}) = \left(x_T + \frac{\hat{x}h}{2},\, y_T + \frac{\hat{y}h}{2}\right)$$

where (x_T, y_T) barycentre of T.

↳ invertible affine map.

$\Rightarrow$ basis functions of $(T, P_T, \Sigma_T) \sim (\hat{T}, \hat{P}, \hat{\Sigma}/\hat{\Sigma}')$

$$p_1(x,y) = \frac{1}{4} + \frac{x - x_T}{h} + \frac{4}{Kh^2}\left[(x-x_T)^2 + (y-y_T)^2\right] \text{ etc.}$$

$$p_1 = \hat{p}_1 \circ F_T^{-1}$$

$$X_h = \{ v_h \in L^2(\Omega) : v_h|_T \in P_T \ \forall T \in \mathcal{T}_h,\ [\![v_h]\!]_E(c_E) = 0 \ \forall E \in \mathcal{E}_h^i \}$$

or $\int_E [\![v_h]\!]_E \, ds = 0$

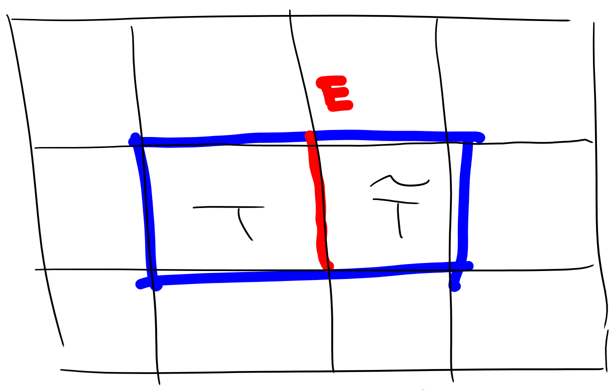
where c_E midpoint of edge E and $\mathcal{E}_h^i$ is set of all interior edges.

$\Sigma_T = \{ v(c_E) : E \subset \partial T \}$ or $\Sigma_T' = \left\{ \frac{1}{|E|} \int_E v\, ds : E \subset \partial T \right\}$

$\Sigma_h = \{$ values at midpoints of edges of $\mathcal{T}_h \}$

or $\Sigma_h' = \left\{ \frac{1}{|E|} \int_E v\, ds,\ E \in \mathcal{E}_h \right\}$ where $\mathcal{E}_h$ set of all edges.

Basis functions of X_h assigned to edges, denoted by φ_E.

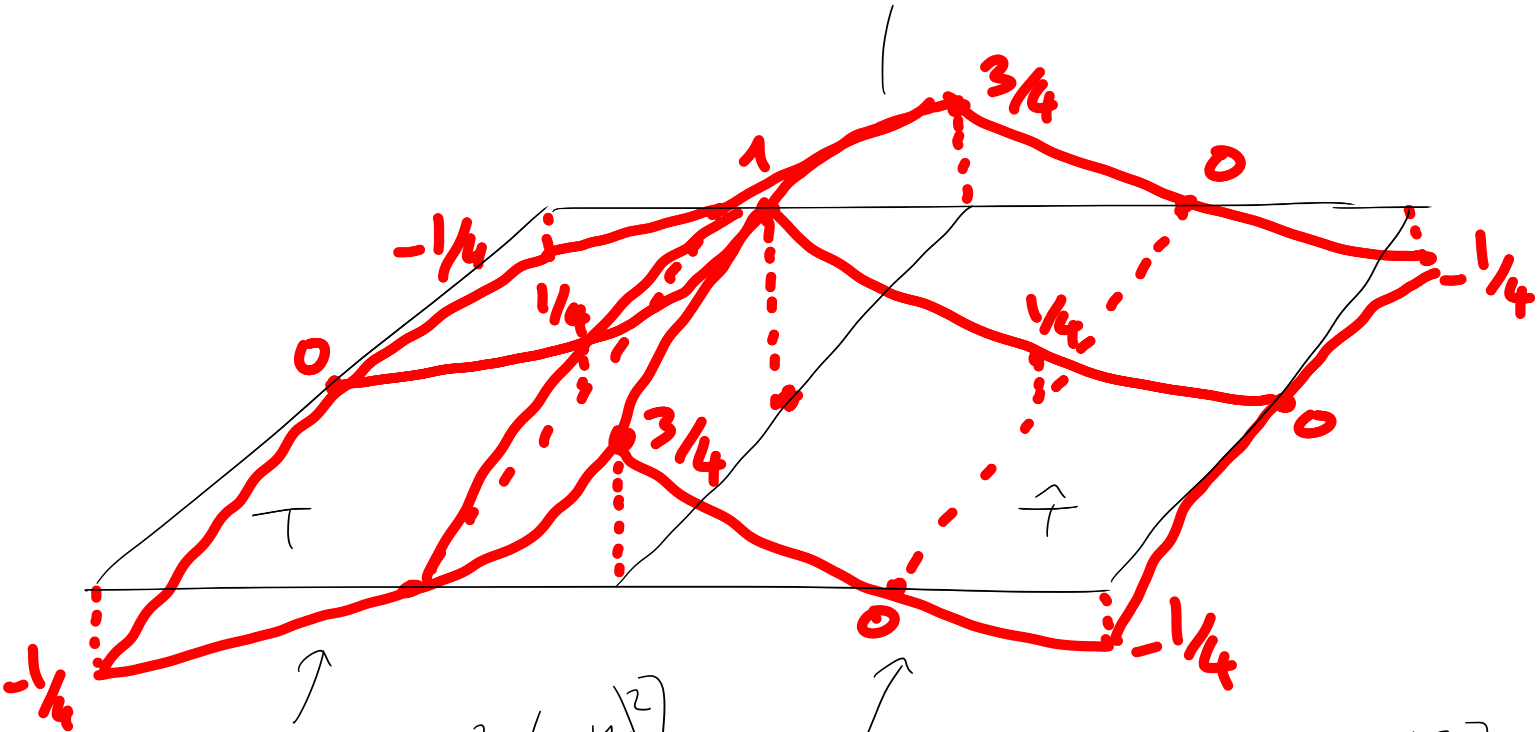


supp φ_E

using midpoint DoFs

$k = 4$

$$1 - \left(\frac{y - y^T}{h} \right)^2$$



$$\frac{1}{4} + \frac{x - x_T}{h} + \left[\left(\frac{x - x_T}{h} \right)^2 + \left(\frac{y - y_T}{h} \right)^2 \right] \qquad \frac{1}{4} - \frac{x - \hat{x}_T}{h} + \left[\left(\frac{x - \hat{x}_T}{h} \right)^2 - \left(\frac{y - \hat{y}_T}{h} \right)^2 \right]$$

Functions in X_h are discontinuous across edges of $\mathcal{T}_h$.