Theorem 2 Let G and $\hat{G}$ be two bounded domains in $\mathbb{R}^n$ with Lipschitz-continuous boundaries which are affine-equivalent, i.e., $G = F(\hat{G})$, where F is an invertible affine mapping $(F(\hat{x}) = \mathbb{B}\hat{x} + b)$.
Let $m \in \mathbb{N}_0$ and $p \in [1, \infty]$. Then, for any $v \in W^{m,p}(G)$, the function $\hat{v} = v \circ F$ belongs to $W^{m,p}(\hat{G})$. Additionally

(6) $|\hat{v}|_{m,p,\hat{G}} \leq C \|\mathbb{B}\|^m |\det \mathbb{B}|^{-1/p} |v|_{m,p,G} \quad \forall v \in W^{m,p}(G)$

(7) $|v|_{m,p,G} \leq C \|\mathbb{B}^{-1}\|^m |\det \mathbb{B}|^{\frac{1}{p}} |\hat{v}|_{m,p,\hat{G}} \quad \forall \hat{v} \in W^{m,p}(\hat{G})$

where C depends only on m & n and $\|\mathbb{B}\| = \sup_{x \in \mathbb{R}^n} \frac{|\mathbb{B}x|}{|x|}$.

Proof Let $v \in C^m(\overline{G})$. Then $\hat{v} \in C^m(\overline{\hat{G}})$ and, for any $i \in \{1, \dots, n\}$ and $\hat{x} \in \hat{G}$ we have that

$$\frac{\partial \hat{v}}{\partial \hat{x}_i}(\hat{x}) = \frac{\partial}{\partial \hat{x}_i} v(F(\hat{x})) = \sum_{k=1}^{n} \frac{\partial v}{\partial x_k}(F(\hat{x})) \underbrace{\frac{\partial F_k(\hat{x})}{\partial \hat{x}_i}}_{B_{ki}}$$

Thus, for $i_1, \dots, i_m \in \{1, \dots, n\}$

$$\left| \frac{\partial^m \hat{v}}{\partial \hat{x}_{i_1} \partial \hat{x}_{i_2} \cdots \partial \hat{x}_{i_m}}(\hat{x}) \right| = \left| \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_m=1}^{n} \frac{\partial^m v}{\partial x_{k_1} \cdots \partial x_{k_m}}(F(\hat{x})) B_{k_1 i_1} \cdots B_{k_m i_m} \right|$$

$$\leq \max_{|\alpha|=m} |(D^\alpha v)(F(\hat{x}))| \left(\left(\sum_{k_1=1}^{n} |B_{k_1 i_1}| \right) \left(\sum_{k_2=1}^{n} |B_{k_2 i_2}| \right) \cdots \left(\sum_{k_m=1}^{n} |B_{k_m i_m}| \right) \right)$$

$$\leq \max_{|\alpha|=m} |(D^\alpha v)(F(\hat{x}))| \left(\sum_{i,j=1}^{n} |B_{ij}| \right)$$

$$\leq C(m,n) \max_{|\alpha|=m} |(D^\alpha v)(F(\hat{x}))| \, \|\mathbb{B}\|^m$$

For any $p \in [1, \infty)$

$$\left| \frac{\partial^m \hat{v}}{\partial \hat{x}_{i_1} \cdots \partial \hat{x}_{i_m}}(\hat{x}) \right|^p \leq C(m,n)^p \max_{|\alpha|=m} |(D^\alpha v)(F(\hat{x}))|^p \|\mathbb{B}\|^{pm}$$

$$\leq C(m,n)^p \|\mathbb{B}\|^{pm} \sum_{|\alpha|=m} |(D^\alpha v)(F(\hat{x}))|^p$$

Then

$$|\hat{v}|_{m,p,\hat{G}} = \left[\int_{\hat{G}} \sum_{|\beta|=m} |\hat{D}^{\beta}\hat{v}(\hat{x})|^p d\hat{x}\right]^{1/p}$$

$$\leq C(m,n)\|B\|^m \left[\int_{\hat{G}} \sum_{|\beta|=m} \sum_{|\alpha|=m} |(D^{\alpha}v)(F(\hat{x}))|^p d\hat{x}\right]^{1/p}$$

$$= \tilde{C}(m,n)\|B\|^m \left[\int_{G} \sum_{|\alpha|=m} |(D^{\alpha}v)(x)|^p \underbrace{\left|\det \frac{DF^{-1}}{Dx}(x)\right|}_{|\det B^{-1}| = |\det B|^{-1}} dx\right]^{1/p}$$

$$= \tilde{C}(m,n)\|B\|^m |\det B|^{-1/p} |v|_{m,p,G}$$

We have proven that

$$|\hat{v}|_{m,p,\hat{G}} \leq \tilde{C}\|B\|^m |\det B|^{-1/p} |v|_{m,p,G} \quad \forall v \in C^m(\overline{G})$$

Now prove for $v \in W^{m,p}(G)$.

Define operator $r : C^m(\overline{G}) \to W^{m,p}(\hat{G})$ by
$rv := v \circ F$. Then, $\|rv\|_{m,p,\hat{G}} \leq C\|v\|_{m,p,G}$.

Since $C^m(\overline{G})$ is dense in $W^{m,p}(G)$ there is a unique $R \in \mathcal{L}(W^{m,p}(G), W^{m,p}(\hat{G}))$ such that $Rv = rv \ \forall v \in C^m(\overline{G})$.

If $v \in W^{m,p}(\Omega)$ then $\exists \{v_\ell\} \subset C^m(\overline{G})$ s.t. $\|v - v_\ell\|_{m,p,G} \to 0$

$$\Rightarrow \|Rv - v\circ F\|_{0,p,\hat{G}} \leq \|R(v - v_\ell)\|_{0,p,\hat{G}} + \|(v_\ell - v)\circ F\|_{0,p,\hat{G}}$$

$(Rv_\ell = r v_\ell = v_\ell \circ F)$

$$\leq C\|v - v_\ell\|_{m,p,\hat{G}} \to 0 \Rightarrow Rv = v\circ F$$

$$\Rightarrow v\circ F \in W^{m,p}(\hat{G})$$

Moreover,

$$|v|_{m,p,\hat{G}} = |Rv|_{m,p,\hat{G}} = \lim_{\ell\to\infty} |\underbrace{Rv_\ell}_{rv_\ell = v_\ell\circ F = \hat{v}_\ell \in C^m(\overline{G})}|_{m,p,\hat{G}}$$

$$\leq \tilde{C}(m,n)\|B\|^m |\det B|^{-1/p} \underbrace{\lim_{\ell\to\infty} |v_\ell|_{m,p,G}}_{|v|_{m,p,G}}$$

$\Rightarrow$ Inequality (6) holds for any $p \in [1,\infty)$.
Need to show (6) for $p=\infty$. If $v \in W^{m,\infty}(G)$
then $v \in W^{m,p}(G)$ $\forall p \in [1,\infty]$ since G is bounded

$\Rightarrow \hat{v} \in W^{m,p}(\hat{G})$ $\forall p \in [1,\infty)$.

Since $\|v\|_{m,p,G} = \left[\sum_{|\alpha|\le m} \int_G |D^\alpha v|^p dx\right]^{1/p} \le \|v\|_{m,\infty,\Omega} \left[\mathrm{card}\{|\alpha|\le m\} |G|\right]^{1/p}$

$$\le C(m,n)\|B\|^{|\alpha|} |\det B|^{-1/p} \|v\|_{m,p,G} \le \tilde{C}(m,n,G,B)\|v\|_{m,\infty,G}$$

Since RHS is independent of p, have that $D^\alpha \hat{v} \in L^\infty(\hat{G})$ $\forall |\alpha| \le m$

$\Rightarrow \hat{v} \in W^{m,\infty}(\hat{G})$

Since $\forall w \in L^\infty(G)$: $\|w\|_{0,\infty,G} = \lim_{p\to\infty} \|w\|_{0,p,G}$

and by (6) $|\hat{v}|_{m,p,\hat{G}} \le C\|B\|^m |\det B|^{-1/p} |v|_{m,p,G}$ $\forall p \in [1,\infty)$,

for $v \in W^{m,\infty}(G)$ we also get 6 for $p=\infty$ (by letting $p\to\infty$).
Inequality (7) follows by interchanging G & $\hat{G}$
since $\hat{G} = F^{-1}(G)$, $\frac{DF^{-1}}{Dx} = B^{-1}$ $\square$

<u>Theorem 5</u> Let $(\hat{T},\hat{P},\hat{\Sigma})$ be a finite element and let
$k,m \in \mathbb{N}_0$ and $p,q \in [1,\infty]$ be such that

$*$ $\hat{\Sigma} \subset [W^{k+1,p}(\hat{T})]'$, $W^{k+1,p}(\hat{T}) \hookrightarrow W^{m,q}(\hat{T})$, condition only on $k,p,$ & q (sob. emb.)

$P_k(\hat{T}) \subset \hat{P} \subset W^{m,q}(\hat{T})$ — In all examples $\hat{P}$ polynomial subspace, which contains polynomial spa & polynomial smooth $\Rightarrow P \subset W^{m,q}(T)$

Then, there exists a constant $\hat{C}$ such that for any finite element (T,P_T,Σ_T) which is affine equivalent to $(\hat{T},\hat{P},\hat{\Sigma})$

$$|v-\Pi_T v|_{m,q,T} \le \hat{C}|T|^{\frac{1}{q}-\frac{1}{p}} \frac{h_T^{k+1}}{\rho_T^m} |v|_{k+1,p,T} \quad \forall v \in W^{k+1,p}(T)$$

where $\Pi_T v$ is the P-interpolation of v.

Lets discuss validity of $*$ $\hat{\Sigma} \subset [W^{k+1,p}(\hat{T})]'$

<u>Lagrange FEs</u> $\hat{\phi} \in \hat{\Sigma} \Rightarrow \exists \hat{z} \in \hat{T} : \hat{\phi}(\hat{v}) = \hat{v}(\hat{z}) \; \forall \hat{v} \in C(\hat{T})$

$\Rightarrow |\hat{\phi}(\hat{v})| \leq \|\hat{v}\|_{0,\infty,\hat{T}} \quad \forall \hat{v} \in C(\hat{T})$

Need Sobolev embedding

Let $\frac{n}{p} < k+1$ (for $n=2,3$ & $p=2$ this holds $\forall k \geq 1$)

Then, $W^{k+1,p}(\hat{T}) \hookrightarrow C(\hat{T})$

$\Rightarrow \|\hat{v}\|_{0,\infty,\hat{T}} \leq C \|\hat{v}\|_{k+1,p,\hat{T}} \quad \forall \hat{v} \in W^{k+1,p}(\hat{T})$

$\Rightarrow |\hat{\phi}(\hat{v})| \leq C \|\hat{v}\|_{k+1,p,\hat{T}} \quad \forall \hat{v} \in W^{k+1,p}(\hat{T})$

$\Rightarrow \hat{\phi} \in [W^{k+1,p}(\hat{T})]' \Rightarrow *$ holds.

<u>Hermite FEs</u> functionals from $\hat{\Sigma}$ are defined on $C^s(\hat{T})$,

for some $s > 0 \Rightarrow |\hat{\phi}(\hat{v})| \leq C \|\hat{v}\|_{s,\infty,\hat{T}} \quad \forall \hat{v} \in C^s(\hat{T}), \hat{\phi} \in \hat{\Sigma}$

If $\frac{n}{p} < k+1-s$, then $W^{k+1,p}(\hat{T}) \hookrightarrow C^s(\hat{T})$

$\Rightarrow \|\hat{v}\|_{s,\infty,\hat{T}} \leq C \|\hat{v}\|_{k+1,p,\hat{T}} \quad \forall \hat{v} \in W^{k+1,p}(\hat{T}) \Rightarrow *$

Functionals from $\hat{\Sigma}$ can be defined using integrals, e.g.

$\phi(\hat{v}) = \frac{1}{|\hat{T}|} \int_{\hat{T}} \hat{v} \, d\hat{x}$. Then $\hat{\phi} \in [L^1(\hat{T})]'$.

If $\hat{T}$ is a triangle & $\hat{E}$ is an edge of $\hat{T}$, then one can define $\hat{\phi}(\hat{v}) = \frac{1}{|\hat{E}|} \int_{\hat{E}} \hat{v} \, ds$. Then,

$|\hat{\phi}(\hat{v})| \leq \hat{C} \|\hat{v}\|_{0,\hat{E}} \leq C \|\hat{v}\|_{1,\hat{T}} \quad \forall \hat{v} \in H^1(\hat{T})$

$\Rightarrow \hat{\phi} \in [H^1(\hat{T})]'$

Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be Lagrange or Hermite FEs, then, $P_k(\hat{T}) \subset \hat{P}$ for some $k \geq 1$. Let $p = q = 2$. Let $\frac{n}{2} < k+1-s$ ($s=0$ for Lagrange) i.e. $n < 2(k+1-s)$ & $0 \leq m \leq k+1$

$[P_2 \subset P_3' \subset P_3 \quad P_3 \subset Q_3' \subset P_{3n}]$

Consider regular triangulations $\frac{h_T}{\rho_T} \leq \sigma \ \forall T$

$\Rightarrow$ for any $(T, P_T, \Sigma_T) \sim (\hat{T}, \hat{P}, \hat{\Sigma})$ $\quad \| v - \Pi_T v \|_{m,T} \leq \hat{C} h_T^{k+1-m} |v|_{k+1,T}$

$\|v - \Pi_T v\|_{m,T}$	$O(h_T^{2-m})$ $0 \leq m \leq 2$ $(k=1)$	$O(h_T^{3-m})$ $0 \leq m \leq 3$ $(k=2)$		$O(h_T^{4-m})$ $0 \leq m \leq 4$ $(k=3)$		
Regularity of v	$H^2(T)$	$H^3(T)$		$H^4(T)$		
Upper bound on n s.t. $H^{k+1}(T) \hookrightarrow C^0(\bar{\Omega})$	$n \leq 3$ $(s=0)$	$n \leq 5$ $(s=0)$	$n \leq 3$ $(s=1)$	$n \leq 7$ $(s=0)$	$n \leq 5$ $(s=1)$	$n \leq 3$ $(s=2)$
Simplex FE	P_1-Lagrange	P_2-Lagrange, Reduced P_3	P_2 Hermite	P_3 Lagrange	P_3 Hermite	
Rectangle FE	Q_1 Lagrange	Q_2 Lagrange, Reduced Q_2		Q_3 Lagrange Reduced Q_3		Bogner-Fox-Schmidt

Inverse Inequality

Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be a finite element & let $l, m \in \mathbb{N}_0$ and $r, q \in [1, \infty]$ be such that

$$l \le m \quad \& \quad \hat{P} \subset W^{l,r}(\hat{T}) \cap W^{m,q}(\hat{T}).$$

Let $\mathcal{T}_h$ be a triangulation and let (T, P_T, Σ_T) be a finite element which is affine equivalent to $(\hat{T}, \hat{P}, \hat{\Sigma})$ for any $T \in \mathcal{T}_h$. Then, there exists a positive constant C_1 depending only on $\hat{T}, \hat{P}, l, m, r, q, n$ such that

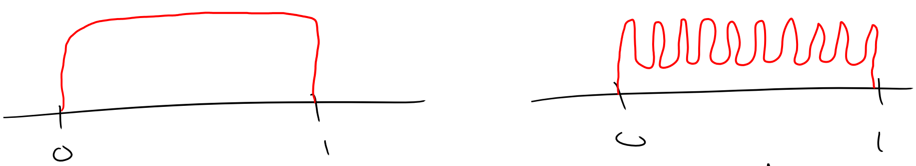
$$|v|_{m,q,T} \le C \frac{h_T^l}{\rho_T^m} |T|^{\frac{1}{q} - \frac{1}{r}} |v|_{l,r,T} \quad \forall v \in P_T \ \forall T \in \mathcal{T}_h.$$

Examples $q = r = 2$

$l = 0, m = 1$: $|v|_{1,T} \le C \frac{1}{\rho_T} \|v\|_{0,T}$ $\forall v \in P_T$

(Friedrich's ineq. $\|v\|_{0,T} \le C |v|_{1,T}$ $\forall v \in H_0^1(T)$)

Only holds in finite dimensional space; counterexample H_0^1:



H^1-seminorm of second greater but L^2 equivalent.

$l = 1, m = 2$: $|v|_{2,T} \le C \frac{h_T}{\rho_T} |v|_{1,T}$ $\forall v \in P_T$

Proof $F_T(\hat{x}) = B\hat{x} + b$ $\quad P_T = \{\hat{v} \circ F_T^{-1} : \hat{v} \in \hat{P}\}$

Consider $v \in P_T$ & $\hat{v} = v \circ F_T$. Then $\hat{v} \in \hat{P}$ & $v \in W^{l,r}(T) \cap W^{m,q}(T)$ and $\hat{v} \in W^{l,r}(\hat{T}) \cap W^{m,q}(\hat{T})$.

We have that $|v|_{m,q,T} \le C \|B^{-1}\|^m |\det B|^{\frac{1}{q}} |\hat{v}|_{m,q,\hat{T}}$.

Equivalence of norms of finite dimensional spaces

$$|\hat{v}|_{m,q,\hat{T}} \le \|\hat{v}\|_{m,q,\hat{T}} \le \hat{C}\|\hat{v}\|_{l,r,\hat{T}}$$

We know that $\inf_{p \in P_{l-1}(\hat{T})} \|\hat{v}+\hat{p}\|_{l,r,\hat{T}} \le \tilde{C}|\hat{v}|_{l,r,\hat{T}}$

Since $l \le m$ we have that $|\hat{v}+\hat{p}|_{m,q,\hat{T}} = |\hat{v}|_{m,q,\hat{T}} \quad \forall \hat{p} \in P_{l-1}(\hat{T})$

$$|\hat{v}|_{m,q,\hat{T}} = |\hat{v}+\hat{p}|_{m,q,\hat{T}} \le \|\hat{v}+\hat{p}\|_{m,q,\hat{T}} \le \hat{C}\|\hat{v}+\hat{p}\|_{l,r,\hat{T}} \quad \forall \hat{p} \in P_{l-1}(\hat{T})$$

$$\Rightarrow |\hat{v}|_{m,q,\hat{T}} \le \hat{C} \inf_{\hat{p} \in P_{l-1}(\hat{T})} \|\hat{v}+\hat{p}\|_{l,r,\hat{T}} \le \bar{C}|\hat{v}|_{l,r,\hat{T}}$$

Additionally, we have that

$$|v|_{m,q,T} \le C\|B^{-1}\|^m |\det B|^{\frac{1}{q}} |\hat{v}|_{m,q,\hat{T}}$$

$$|\hat{v}|_{l,r,\hat{T}} \le C\|B\|^l |\det B|^{-\frac{1}{r}} |v|_{l,r,T}$$

So $|v|_{m,q,T} \le C\|B\|^l \|B^{-1}\|^m |\det B|^{\frac{1}{q}-\frac{1}{p}} |v|_{l,r,T}$

We hav that

$$\|B\| \le \frac{h_T}{\hat{\rho}} \qquad \|B^{-1}\| \le \frac{\hat{h}}{\rho} \qquad |\det B| = \frac{|T|}{|\hat{T}|}$$

$$\Rightarrow |v|_{m,q,T} \le C\frac{h_T^l}{\rho_T^m}|T|^{\frac{1}{q}-\frac{1}{p}}|v|_{l,r,T}$$

$\square$