Consider derivation of errors bound for an example:

- Consider finite element (T, P_T, Σ_T):
  - T triangle with vertices a_1, a_2, a_3
  - $\lambda_1, \lambda_2, \lambda_3$ barycentric coordinates w.r.t vertices
  - $P_T = \text{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2\lambda_3\}$
  - $\Sigma_T = \{\phi_i\}_{i=1}^{4}$ where

$$\phi_i(v) = v(a_i) \qquad i = 1,2,3$$

$$\phi_4(v) = \frac{1}{|T|}\int_T v\,dx$$

for any $v \in C(T)$.

Verify this is a finite element - only consider unisolvence:

Consider any $\alpha_1, \dots, \alpha_4 \in \mathbb{R}$, want to show $\exists$ unique function $p \in P_T$ such that

$$\phi_i(p) = \alpha_i \quad i = 1, \dots, 4$$

Since $p = \sum_{j=1}^{3} \beta_j \lambda_j + \beta_4 \lambda_1\lambda_2\lambda_3$ for some $\beta_1, \dots, \beta_4$ we get for $i = 1, \dots, 3$ $\alpha_i = \phi_i(p) = p(a_i) = \beta_i$ & β_4 uniquely determined by

$$\alpha_4 = \phi_4(p) = \sum_{j=1}^{3} \alpha_j \phi_4(\lambda_j) + \beta_4 \phi_4(\lambda_1\lambda_2\lambda_3)$$

Define the space

$$X_h = \{v \in L^2(\Omega) : v|_T \in P_T \ \forall T \in \mathcal{T}_h, \ v|_T(z_i) = v|_{\tilde{T}}(z_i) \ \forall T \in \mathcal{T}_{h,i}, i = 1 \dots N\}$$

where $z_1, \dots, z_N$ are vertices of $\mathcal{T}_h$ & $\mathcal{T}_{h,i} = \{T \in \mathcal{T}_h, T \ni z_i\}$

$$\Rightarrow \dim X_h = N + \text{card}\,\mathcal{T}_h$$

Basis functions of X_h are functions $\{p_i\}_{i=1}^N$ assigned to vertices and functions $\{p_T\}_{T\in\mathcal{T}_h}$ assigned to elements of $\mathcal{T}_h$.

$$\operatorname{supp} p_i = \bigcup_{T\ni z_i} T \qquad \operatorname{supp} p_T = T$$

$$p_i(z) = \delta_{ij} \quad \text{and} \quad \int_T p_i\,dx = 0 \qquad i=1,\dots,N,\ T\in\mathcal{T}_h$$

$$p_T = \lambda_1\lambda_2\lambda_3 \quad \text{in } T$$

Can show $v_h \in C(\bar{\Omega})$ for any $v_h \in V_h$.

Denote degrees of freedom in Σ_h by

$$\phi_i(v) = v(z_i) \quad i=1,\dots,N \qquad \phi_T(v) = \frac{1}{|T|}\int_T v\,dx \quad \forall T\in\mathcal{T}_h$$

Interpolation operator defined by $\forall v\in C(\bar{\Omega})$:

$$\Pi_h v \in X_h \qquad \phi_i(\Pi_h v) = \phi_i(v) \qquad i=1,\dots,N$$

$$\phi_T(\Pi_h v) = \phi_T(v) \qquad T\in\mathcal{T}_h$$

We have that $(\Pi_h v)|_T = \Pi_T(v|_T)$, where Π_T is P_T-interpolation operator defined by conditions

$$\forall v\in C(T):\ \Pi_T v \in P_T:\ \phi_i(\Pi_T v) = \phi_i(v),\ i=1,\dots,4$$

where

$$\phi_i(v) = v(a_i) \quad i=1,2,3 \qquad \phi_4(v) = \frac{1}{|T|}\int_T v\,dx$$

Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be reference element. Then,

$$\widehat{\Pi_T v} = \hat{\Pi}\hat{v} \qquad \forall v\in C(T),\ T\in\mathcal{T}_h$$

Denote by $\hat{p}_1,\dots,\hat{p}_4$ basis functions of $(\hat{T}, \hat{P}, \hat{\Sigma})$, then

$$\hat{\Pi}\hat{v} = \sum_{i=1}^{4} \hat{\phi}_i(\hat{v})\,\hat{p}_i$$

Consider $\hat{v} \in H^2(\Omega)$, since $H^2(\hat{T}) \hookrightarrow C(\hat{T})$

$$|\hat{\Phi}_i(\hat{v})| = |\hat{v}(\hat{a}_i)| \leq \|\hat{v}\|_{0,\infty,\hat{T}} \leq C\|\hat{v}\|_{2,\hat{T}} \quad i=1,2,3$$

$$|\hat{\Phi}_4(\hat{v})| \leq \frac{1}{\sqrt{|\hat{T}|}} \|u\|_{0,\hat{T}} \leq \frac{1}{\sqrt{|\hat{T}|}} \|\hat{v}\|_{2,\hat{T}}$$

Thus, for any norm $\|\circ\|$ on $\hat{P}$ we have

$$\|\hat{\Pi}\hat{v}\| \leq \sum_{i=1}^{4} |\hat{\Phi}_i(\hat{v})| \cdot \|\hat{p}_i\| \leq C\|\hat{v}\|_{2,\hat{T}} \quad \forall \hat{v} \in H^2(\hat{T})$$

We also use fact that $\hat{\Pi}\hat{v} = \hat{v} \quad \forall \hat{v} \in \hat{P}$

For any $T \in \mathcal{T}_h$ and $m \in \{0,1\}$

$$\begin{aligned}
|v - \Pi_T v|_{m,T} &\leq C h_T^{1-m} |\widehat{v - \Pi_T v}|_{m,\hat{T}} \\
&= C h_T^{1-m} \inf_{\hat{q} \in P_1(\hat{T})} |\hat{v} + \hat{q} - \Pi(\hat{v} + \hat{q})|_{m,\hat{T}} \\
&\leq \tilde{C} h_T^{1-m} \inf_{\hat{q} \in P_1(\hat{T})} \|\hat{v} + \hat{q}\|_{2,\hat{T}} \\
&\leq \bar{C} h_T^{1-m} |\hat{v}|_{2,\hat{T}} \\
&\leq \hat{C} h_T^{2-m} |v|_{2,T} \qquad \forall v \in H^2(T)
\end{aligned}$$

Thus, for $v \in H^2(\Omega)$

$$\|v - \Pi v\|_{0,\Omega} \leq C h^2 |v|_{2,\Omega} \qquad |v - \Pi v|_{1,\Omega} \leq C h |v|_{2,\Omega}$$

We now consider a finite element formulation on the space X_h for Poisson equation in Ω with homogeneous Dirichlet BCs:

$$-\Delta u = f \text{ in } \Omega \qquad u = 0 \text{ on } \partial\Omega$$

and prove error estimates w.r.t H^1 & L^2.

Weak formulation reads: find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$$

where $a(u,v) = \int_\Omega \nabla u \nabla v \, dx$

The discrete problem reads

find $u_h \in V_h = \{v_h \in X_h : v_h = 0 \text{ on } \partial\Omega\}$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Since $V_h \subset H_0^1(\Omega)$ we have that

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

Hence,

$$|u-u_h|_{1,\Omega}^2 = a(u-u_h, u-u_h)$$
$$= a(u-u_h, u - \underbrace{\Pi_h u}_{\in V_h})$$
$$\leq |u-u_h|_{1,\Omega} |u-\Pi_h u|_{1,\Omega}$$

$$\Rightarrow \|u-u_h\|_{1,\Omega} \leq C|u-u_h|_{1,\Omega} \leq C|u-\Pi_h u|_{1,\Omega}$$
$$\leq Ch|u|_{2,\Omega}$$

(Error in H^1-norm).

We used fact that problem is regular in sense that
$\forall f \in L^2(\Omega)$ $u \in H^2(\Omega)$ $\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$

To prove an estimate in L^2-norm we denote by $\varphi \in H_0^1(\Omega)$ solution of

$$a(v, \varphi) = (u-u_h, v) \quad \forall v \in H_0^1(\Omega)$$

Regularity implies that $\varphi \in H^2(\Omega)$ and $\|\varphi\|_{2,\Omega} \le C\|u-u_h\|_{0,\Omega}$. Since $a(u-u_h, \Pi_h\varphi) = 0$

$$\|u-u_h\|_{0,\Omega}^2 = (u-u_h, \underbrace{u-u_h}_{\in H_0^1(\Omega)})$$
$$= a(u-u_h, \varphi)$$
$$= a(u-u_h, \varphi - \Pi_h\varphi)$$
$$\le C h |u-u_h|_{1,\Omega} |\varphi - \Pi_h\varphi|_{1,\Omega}$$
$$\le \tilde{C} h |u-u_h|_{1,\Omega} |\varphi|_{2,\Omega}$$
$$\le \tilde{C} h |u-u_h|_{1,\Omega} \|u-u_h\|_{0,\Omega}$$

$$\Rightarrow \|u-u_h\|_{0,\Omega}^2 \le \tilde{C} h |u-u_h|_{1,\Omega} \le C h^2 |u|_{2,\Omega}$$