Variational Crimes Ex 1:

Let $\Omega \subset \mathbb{R}^2$ be bounded domain with polygonal boundary, and $\mathcal{T}_h$ its triangulation consisting of triangles. Consider the weak formulation: find $u \in H_0^1(\Omega)$ such that

$$\sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

where $a_{ij}, f \in C(\overline{\Omega})$, $\exists \Theta > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \geq \Theta |\xi|^2 \quad \forall \xi \in \mathbb{R}^2, x \in \overline{\Omega}.$$

Let $V_h \subset H_0^1(\Omega)$ be the finite element space of continuous piecewise linear functions and define the discrete problem: find $u_h \in V_h$ such that

$$\sum_{i,j=1}^{2} \sum_{T \in \mathcal{T}_h} \int_T a_{ij}(C_T) \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} dx = \sum_{T \in \mathcal{T}_h} \int_T f(C_T) v_h dx \quad \forall v_h \in V_h.$$

where C_T is the barycentre of the triangle T.

Show that, under suitable assumptions

$$|u - u_h|_{1,\Omega} \leq Ch$$

where C does not depend on h & specify suffizient conditions for this.

Proof

Define the operator $\Pi_T : C(T) \to C(T)$ by $\Pi_T v = v - v(c_T)$. Then, $\Pi_T v = 0 \quad \forall v \in P_0(T)$.

Assuming shape regularity

$$\|\Pi_T v\|_{0,\infty,T} = \|\Pi_{\hat T}\hat v\|_{0,\infty,\hat T}$$

$$= \inf_{\hat q \in P_0(\hat T)} \|\Pi_{\hat T}(\hat v - \hat q)\|_{0,\infty,\hat T}$$

$$\le 2 \inf_{\hat q \in P_0(\hat T)} \|\hat v - \hat q\|_{0,\infty,\hat T}$$

$$\le \bar C |\hat v|_{1,\infty,\hat T} \le C h_T |v|_{1,\infty,T} \quad \forall v \in W^{1,\infty}(T)$$

Define

$$a(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

$$a_h(u,v) = \sum_{T \in T_h} \sum_{i,j=1}^{2} \int_T a_{ij}(c_T) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

$$\langle f, v \rangle = \int_\Omega f v \, dx \qquad \langle f_h, v_h \rangle = \sum_{T \in T_h} \int_T f(c_T) v \, dx$$

Then, if $a_{ij} \in W^{1,\infty}(\Omega)$,

$$|a(v_h,w_h) - a_h(v_h,w_h)| = \left| \sum_{T \in T_h} \int_T \sum_{i,j=2}^{2} \overbrace{(a_{ij} - a_{ij}(c_T))}^{= \Pi_T a_{ij}} \frac{\partial v_h}{\partial x_i} \frac{\partial w_h}{\partial x_j} dx \right|$$

$$\le \sum_{T \in T_h} \sum_{i,j=1}^{2} C h_T |a_{ij}|_{1,\infty,T} \int_T \left|\frac{\partial v_h}{\partial x_i}\right| \left|\frac{\partial w_h}{\partial x_j}\right| dx$$

$$\le C h |v_h|_{1,\Omega} |w_h|_{1,\Omega}$$

and if $f \in W^{1,\infty}(\Omega)$

$$|\langle f, w_h \rangle - \langle f_h, w_h \rangle| \le \sum_{T \in T_h} \left| \int_T \overbrace{(f - f(c_T))}^{\Pi_T f} w_h \, dx \right| \le C h^{3/2} \|v_h\|_{0,\Omega}$$

$$\left(\int_T v_h \, dx \le \|1\|_{0,T} \|v_h\|_{0,T} \le C h_T^{1/2} \|v_h\|_{0,T} \right)$$

Therefore,

$$a_h(u_h - v_h, w_h) = \langle f_h, w_h\rangle - \langle f, w_h\rangle + a(v_h, w_h) - a_h(v_h, w_h) + a(u - v_h, w_h)$$

$$\leq Ch^{3/2}|w_h|_{1,\Omega} + Ch|v_h|_{1,\Omega}|w_h|_{1,\Omega} + C|u - v_h|_{1,\Omega}|w_h|_{1,\Omega}$$

$$\Rightarrow |u - u_h|_{1,\Omega} \leq Ch(1 + |v_h|_{1,\Omega}) + C|u - v_h|_{1,\Omega} \quad \forall v_h \in V_h$$

$$\Rightarrow |u - u_h|_{1,\Omega} \leq Ch \quad \text{for } u \in H^2(\Omega) \qquad \square$$

Ex 2:

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary and $\mathcal{T}_h$ its triangulation consisting of triangles. Consider the problem

$$-\nu\Delta u + cu = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $c, f \in C^\infty(\bar{\Omega})$, $c \geq 0$ in Ω. The weak formulation is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nu \nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx = \int_\Omega fv \, dx \quad \forall v \in H_0^1(\Omega)$$

Consider the space $V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in P_k(T) \; \forall T \in \mathcal{T}_h\}$

For $\mathcal{T}_h$ satisfying regularity and compatibility assumptions, the Lagrange interpolation operator Π_h satisfies

$$\|v - \Pi_h v\|_{0,\Omega} + h|v - \Pi_h v|_{1,\Omega} \leq Ch^{m+1}|v|_{m+1,\Omega}$$

$$\forall v \in H^{m+1}(\Omega) \cap H_0^1(\Omega), \; m = 1, \dots, k.$$

For any $T \in \mathcal{T}_h$, define $r_T : C(T) \to P_1(T)$ such that $r_T v = v$ at the vertices of T for any $v \in C(T)$.

Define the discrete problem: find $u_h \in U_h$ such th

$$\nu \int_\Omega \nabla u_h \cdot \nabla v_h \, dx + \sum_{T \in \mathcal{T}_h} \int_T (r_T c) u_h v_h \, dx = \sum_{T \in \mathcal{T}_h} \int_T (r_T f) v_h \, dx$$

$$\forall v_h \in V_h.$$

Investigate convergence order of $|u - u_h|_{1,\Omega}$.

We have that $\|v - r_T v\|_{0,T} \leq C h_T^2 |v|_{2,T} \quad \forall v \in H^2(T)$.

Thus,

$$|a_h(u,v) - a(u,v)| \leq \sum_{T \in \mathcal{T}_h} \left| \int_T (c - r_T c) u v \, dx \right|$$

$$\leq \sum_{T \in \mathcal{T}_h} \|c - r_T c\|_{0,T} \|u\|_{0,4,T} \|v\|_{0,4,T}$$

$$\leq C h^2 |c|_{2,\Omega} \|u\|_{0,4,T} \|v\|_{0,4,T}$$

$$\leq \tilde{C} h^2 |c|_{2,\Omega} |u|_{1,\Omega} |v|_{1,\Omega}$$

$$\forall u, v \in H^1(\Omega)$$

(alternatively, use fact that $\|v - r_T v\|_{0,\infty,T} \leq C h_T^2 |v|_{2,\infty,T}$)

$$|\langle f_h, v\rangle - \langle f, v\rangle| \leq C h^2 |f|_{2,\Omega} |v|_{1,\Omega}$$

Then,

$$a_h(u - u_h, v_h) = a_h(u, v_h) - a(u, v_h) + \langle f, v_h\rangle - \langle f_h, v_h\rangle$$

$$\leq C h^2 |v_h|_{1,\Omega}$$

Thus, $$\nu |u_h - \Pi_h u|_{1,\Omega} \leq \sup_{v \in V_h} \frac{a_h(u_h - \Pi_h u, v_h)}{|v_h|_{1,\Omega}}$$

$$\leq C\left(|u - \Pi_h u|_{1,\Omega} + h^2\right)$$

$$\Rightarrow |u_h - u_h|_{1,2} \leq C h^{\min\{k,2\}}$$

In general,

$$\|u-u_h\|_V \leq \frac{1}{\tilde{\alpha}} \inf_{v_h \in V_h} \left\{ (M+\tilde{\alpha})\|u-v_h\|_V + \sup_{w_h \in V_h} \frac{|a(v_h,w_h) - a_h(v_h,w_h)|}{\|w_h\|_V} \right\}$$

$$+ \frac{1}{\tilde{\alpha}} \sup_{w_h \in V_h} \frac{|\langle f, w_h\rangle - \langle f_h, w_h\rangle|}{\|w_h\|_V}$$