Numerical Solution of ODEs

Exercise Class

10th October 2023

Cat & Mouse

We consider an ordinary differential equation system modeling a cat chasing a mouse (in two dimensions). The mouse moves from an initial position $(1, 0)$ to its hole at $(0, 0)$ with constant velocity v_m. We can model its position $\boldsymbol{y}(t) = (y_1(t), y_2(t))$ at time t as

$$\boldsymbol{y}(t) = \begin{pmatrix} 1 - v_m t \\ 0 \end{pmatrix}, \qquad t \in \left[0, \frac{1}{v_m}\right].$$

The cat moves from an initial position $\widetilde{\boldsymbol{x}} \in \mathbb{R}^2$, $\widetilde{x}_2 > 0$, with constant velocity $v_c > 0$ towards the mouses' *current position*. The position of the cat at time t is denoted by $\boldsymbol{x}(t) = (x_1(t), x_2(t))$. To calculate the cat's position we note that at time t the mouse moves in the direction of the vector $\boldsymbol{y}(t) - \boldsymbol{x}(t)$; i.e., towards the mouse. The change in position of the cat with respect to time is given by

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= f(t)(y_1(t) - x_1(t)) = f(t)(1 - v_m - x_1) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f(t)(y_2(t) - x_2(t)) = -f(t)x_2 \end{aligned}$$

for a function $f(t)$ dependent on t. We can then calculate the velocity of the cat as

$$\left(\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right)^2\right).$$

However, we know the cat has fixed velocity v_c; therefore,

$$\begin{aligned} & v_c^2 = (f(t))^2 \left((1 - v_m t - x_1)^2 + x_2^2\right) \\ \Longrightarrow \qquad & f(t) = v_c \left((1 - v_m t - x_1)^2 + x_2^2\right)^{-1/2} \end{aligned}$$

Hence, we derive the ODE system

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \end{pmatrix} = \frac{v_c}{\left((1 - v_m t - x_1)^2 + x_2^2\right)^{1/2}} \begin{pmatrix} 1 - v_m t - x_1 \\ -x_2 \end{pmatrix}$$

Exercises

1. Play with the cat & mouse simulation `catmouse.m/run_catmouse.m`

   (a) Try various starting positions $\widetilde{\boldsymbol{x}}$ for the cat and velocities v_m, v_c.

   (b) Is there any strange behaviour occuring? When does it occur and why?

   (c) Study and use `run_catmousevents.m` to see how `ode23` can be used to detect an event — the cat catching the mouse ($|\boldsymbol{x} - \boldsymbol{y}| = 0$).

2. Does the cat & mouse simulation work in one dimension; i.e., $\widetilde{x}_2 = 0$? Solve analytically when/if the cat catches the mouse in this situation for *all* possible starting positions of the cat.

3. The built-in MATLAB simulation `ballode` simulates a bouncing ball:

$$x'' = -g \qquad \equiv \qquad \begin{cases} x_1' = x_2 \\ x_2' = -g \end{cases}$$

   where $g = -9.8\,\mathrm{m/s^2}$ is the Earth's gravity, with attenuation of the velocity on hitting the ground (90%). Copy this script and create a variant to handle dynamical friction:

$$x'' = -g - ax'$$

   where $a > 0$ represents friction.

4. Use `blowupex.m` to investigate the blow-up behaviour of the ODE:

$$x'(t) = x^2, \\ x(0) = 1$$

   (a) Plot, using `dirfield.m` from last week, the direction field over the intervals $t \in [-1, 1]$ and $x \in [-2, 2]$.

   (b) Study the solutions, when computing using `ode23`, over the time intervals $t \in [0, 0.99]$, $t \in [0, 1]$, and $t \in [0, 2]$.

5. Use `collapseex.m` to investigate the collapse behaviour of the ODE:

$$x'(t) = -x^{-1/2}, \\ x(0) = 1$$

   (a) Plot, using `dirfield.m` from last week, the direction field over the intervals $t \in [-1, 1]$ and $x \in [.01, 2]$.

   (b) Study the solutions, when computing using `ode23`, over the time intervals $t \in [0, 2/3]$ and $t \in [0, 1]$.