Numerical Solution of ODEs

Exercise Class

17th October 2023

Explicit One-Step Methods

Euler Implemented by `eul.m`:

$$\begin{aligned} \kappa_1 &= f(t,x), \\ \psi(t+\tau,t,x) &= x + \tau\kappa_1. \end{aligned}$$

Runge Implemented by `runge.m`:

$$\begin{aligned} \kappa_1 &= f(t,x), \\ \kappa_2 &= f\left(t+\frac{\tau}{2}, x+\frac{\tau}{2}\kappa_1\right), \\ \psi(t+\tau,t,x) &= x + \tau\kappa_2. \end{aligned}$$

Runge-Kutta Implemented by `rk_classical.m`:

$$\begin{aligned} \kappa_1 &= f(t,x) \\ \kappa_2 &= f\left(t+\frac{\tau}{2}, x+\frac{\tau}{2}\kappa_1\right), \\ \kappa_3 &= f\left(t+\frac{\tau}{2}, x+\frac{\tau}{2}\kappa_2\right), \\ \kappa_4 &= f\left(t+\tau, x+\tau\kappa_3\right), \\ \psi(t+\tau,t,x) &= x + \tau\left(\frac{1}{6}\kappa_1+\frac{1}{3}\kappa_2+\frac{1}{3}\kappa_3+\frac{1}{6}\kappa_4\right). \end{aligned}$$

Heun

$$\begin{aligned} \kappa_1 &= f(t,x), \\ \kappa_2 &= f\left(t+\tau, x+\tau\kappa_1\right), \\ \psi(t+\tau,t,x) &= x + \frac{\tau}{2}\left(\kappa_1+\kappa_2\right). \end{aligned}$$

Exercises

1. Compare the solution obtained by the Euler, Runge, and Runge-Kutta methods with $\tau = {}^1/_2, {}^1/_4, {}^1/_8$ to the solution obtained with `ode23` for the following problems:

   (a) Logistic equation

   $$\begin{aligned} x(t)' &= (a - bx(t))x(t), & t \in [0,3], \\ x(0) &= x_0, \end{aligned}$$

   with $a = b = 1$ and $x_0 = 2$.

(b) The pendulum problem:

$$x''(t) = -k\sin(x(t)),$$
$$x(t_0) = x_0$$

with $k = 1$, $t = (0, 6\pi)$, and various initial conditions

$$x_0 = \begin{pmatrix} -1.5 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -\pi \\ 1 \end{pmatrix}.$$

(c) The harmonic oscillator

$$x''(t) + bx = c\cos(\omega t),$$
$$x(t_0) = x_0$$

with

- $a = 0$, $b = 9$, $c = 10$
- $t = [0, 50]$
- $x_0 = (1, 0)^\top$
- $\omega = 2.5, 2.9, 3.1, 3, \sqrt{3}$

2. Implement the Heun method as a MATLAB function, and test with the logistic equation with $\tau = 1/2, 1/4, 1/8$.