Numerical Solution of ODEs

Exercise Class

21st November 2023

Exercises

1. Derive the recursive formula for the following methods using the numerical integration definition

$$u(t_{j+1}) \approx u(t_{j-k}) + \sum_{i=0}^{q+\ell} f_{j-q+i} \int_{t_{j-k}}^{t^{j+1}} \mathcal{L}_{j-q+i}(s)\,\mathrm{d}s,$$

where $f_{j-q+i} = f(t_{j-q+i}, u(t_{j-q+i}))$ and

$$\mathcal{L}_{j-q+i}(s) = \prod_{\substack{k=0 \\ k\neq i}}^{q+\ell} \frac{s - t_{j-q+k}}{t_{j-q+i} - t_{j-q+k}}, \qquad t_{i-q} \leq s \leq t_{j+1},$$

for $i = 0, \dots, q+\ell$ are the Lagrange basis functions.

   (a) 2-step Nyström method ($k = 1$, $\ell = 0$, $q = 1$)

   (b) 2-step Milne-Simpson method ($k = 1$, $\ell = 1$, $q = 1$)

2. Compare the numerical solutions given by the 2-step Milne-Simpson, Nyström, and Adams-Moulton methods, along with the numerical solution from `ode23` for the following ODEs.

   (a) Linear oscillator (`oscillator.m`) on the time interval $t \in [0, 10]$, with $\tau = 0.1$:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -9x + 10\cos(t), \\ \boldsymbol{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

   For comparisons plot t vs. x_1.

   (b) Stiff linear system (`linsystem.m`) on the time interval $t \in [0, 0.05]$ with $\tau = 0.001$:

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

   Also run convergence analysis using `conv_analysis.m` to deduce the order of the *Milne-Simpson*, and *Nyström* 2-step methods.