Recap

$$u_{j+1} \approx u_{j-k} + \sum_{i=0}^{q+l} f_{j-q+i} \int_{t_{j-k}}^{t_{j+1}} \mathcal{L}_{j-q+i}(s)\,ds$$

where

$$\mathcal{L}_{j-q+i}(s) = \prod_{\substack{k=0\\k\neq i}}^{q+l} \frac{s-t_{j-q-k}}{t_{j-q+i}-t_{j-q+k}}$$

Nyström Methods - $k=1, l=0$ (Explizit)

- $q=2 \Rightarrow$ 3-step Nyström methods (ny3om)

$$u_{j+1} = u_{j-1} + \tau\left(\frac{7}{3} f(t_j,u_j) - \frac{2}{3} f(t_{j-1},u_{j-1}) + \frac{1}{2} f(t_{j-2},u_{j-2})\right)$$

- $q=0 \Rightarrow$ 1-step Nyström method

$$u_{j+1} \approx u_{j-1} + \sum_{i=0}^{0} f_{j+i} \int_{t_{j-1}}^{t_{j+1}} \prod_{\substack{k=0\\k\neq i}}^{0} \frac{s-t_{j+i}}{t_{j+i}-t_{j+k}}\,ds$$

$$= u_{j-1} + f_j \int_{t_{j-1}}^{t_{j+1}} ds$$

$$= u_{j-1} + [s]_{t_{j-1}}^{t_{j+1}} f_j$$

$$= u_{j-1} + (t_{j+1}-t_{j-1}) f_j$$

$$= u_{j-1} + 2\tau f_j$$

- $q=1 \Rightarrow$ 2-step Nyström method

$$u_{j+1} \approx u_{j-1} + \sum_{i=0}^{1} f_{j+i-1} \int_{t_{j-1}}^{t_{j+1}} \prod_{\substack{k=0\\k\neq i}}^{1} \frac{s-t_{j+k-1}}{t_{j+i-1}-t_{j+k-1}}\,ds$$

$$= u_{j-1} + f_{j-1} \int_{t_{j-1}}^{t_{j+1}} \frac{s-t_j}{t_{j-1}-t_j}\,ds + f_j \int_{t_{j-1}}^{t_{j+1}} \frac{s-t_{j-1}}{t_j-t_{j-1}}\,ds$$

$$= u_{j-1} + f_{j-1}\left[\frac{\frac{1}{2}s^2 - t_j s}{t_{j-1}-t_j}\right]_{t_{j-1}}^{t_{j+1}} + f_j\left[\frac{\frac{1}{2}s^2 - t_{j-1}s}{t_j - t_{j-1}}\right]_{t_{j-1}}^{t_{j+1}}$$

$$= u_{j-1} - \frac{1}{\tau} f_{j-1}\left(\frac{1}{2}t_{j+1}^2 - t_j t_{j+1} - \frac{1}{2}t_{j-1}^2 - t_j t_{j-1}\right)$$

$$+ \frac{1}{\tau} f_j \left(\frac{1}{2}t_{j+1}^2 - t_{j-1}t_{j+1} - \frac{1}{2}t_{j-1}^2 + t_{j-1}^2\right)$$

$$= u_{j-1} - \frac{1}{\tau} f_{j-1}\left(\frac{1}{2}(t_{j+1}-t_{j-1})(t_{j+1}+t_{j-1}) - t_j(t_{j+1}-t_{j-1})\right)$$

$$+ \frac{1}{\tau} f_j \left(\frac{1}{2}(t_{j+1}-t_{j-1})(t_{j+1}+t_{j-1}) - t_{j-1}(t_{j+1}-t_{j-1})\right)$$

$$= u_{j-1} - \frac{1}{\tau} f_{j-1}\left(\tau(t_{j+1}+t_{j-1}) - 2\tau t_j\right)$$
$$+\frac{1}{\tau} f_j \left(\tau(t_{j+1}+t_{j-1}) - 2\tau t_{j-1}\right)$$
$$= u_{j-1} - f_{j-1}\Big(\underbrace{\underbrace{t_{j+1}-t_j}_{\tau} + \underbrace{t_{j-1}-t_j}_{-\tau}}_{=0}\Big) + f_j \underbrace{(t_{j+1}-t_{j-1})}_{2\tau}$$
$$= u_{j-1} + 2\tau f_j \equiv \text{1-step Nyström!}$$

<u>Milne-Simpson</u> $k=1, l=1$ (Implizit)

$$u_{j+1} \simeq u_{j-1} + \sum_{i=0}^{2} f_{j-1+i} \int_{t_{j-1}}^{t_{j+1}} \prod_{\substack{k=0\\k\neq i}}^{2} \frac{s-t_{j-1+k}}{s_{j-1+i}-t_{j-1+k}}\, ds$$

$$= u_{j-1} + f_{j-1}\int_{t_{j-1}}^{t_{j+1}} \left(\frac{s-t_j}{t_{j-1}-t_j}\right)\left(\frac{s-t_{j+1}}{t_{j-1}-t_{j+1}}\right) ds$$
$$+ f_j \int_{t_{j-1}}^{t_{j+1}} \left(\frac{s-t_{j-1}}{t_j-t_{j-1}}\right)\left(\frac{s-t_{j+1}}{t_j-t_{j+1}}\right) ds$$
$$+ f_{j+1} \int_{t_{j-1}}^{t_{j+1}} \left(\frac{s-t_{j-1}}{t_{j+1}-t_{j-1}}\right)\left(\frac{s-t_j}{t_{j+1}-t_j}\right) ds$$

$$= u_{j-1} + f_{j-1}\frac{1}{2\tau^2}\int_{t_{j-1}}^{t_{j+1}} s^2 - t_j s - t_{j+1}s + t_j t_{j+1}\, ds$$
$$- f_j \frac{1}{\tau^2}\int_{t_{j-1}}^{t_{j+1}} s^2 - t_{j-1}s - t_{j+1}s + t_{j-1}t_{j+1}\, ds$$
$$+ f_{j+1}\frac{1}{2\tau^2}\int_{t_{j-1}}^{t_{j+1}} s^2 - t_{j-1}s - t_j s + t_{j-1}t_j\, ds$$

$$= u_{j-1} + f_{j-1}\frac{1}{2\tau^2}\left[\frac{1}{3}s^3 - \frac{1}{2}(t_j+t_{j+1})s^2 + t_j t_{j+1} s\right]_{t_{j-1}}^{t_{j+1}}$$
$$- f_j \frac{1}{\tau^2}\left[\frac{1}{3}s^3 - \frac{1}{2}(t_{j-1}+t_{j+1})s^2 + t_{j-1}t_{j+1}s\right]_{t_{j-1}}^{t_{j+1}}$$
$$+ f_{j+1}\frac{1}{2\tau^2}\left[\frac{1}{3}s^2 - \frac{1}{2}(t_{j-1}+t_j)s^2 + t_{j-1}t_j s\right]_{t_{j-1}}^{t_{j+1}}$$

$$= u_{j-1} + f_{j-1}\frac{1}{2\tau^2}\left[\frac{1}{3}(t_{j+1}^3 - t_{j-1}^3) - \frac{1}{2}(t_j+t_{j+1})(t_{j+1}^2-t_{j-1}^2) + t_j t_{j+1}(t_{j+1}-t_{j-1})\right]$$
$$- f_j \frac{1}{\tau^2}\left[\frac{1}{3}(t_{j+1}^3 - t_{j-1}^3) - \frac{1}{2}(t_{j-1}+t_{j+1})(t_{j+1}^2-t_{j-1}^2) + t_{j-1}t_{j+1}(t_{j+1}-t_{j-1})\right]$$
$$+ f_{j+1}\frac{1}{2\tau^2}\left[\frac{1}{3}(t_{j+1}^3 - t_{j-1}^3) - \frac{1}{2}(t_{j-1}+t_j)(t_{j+1}^2-t_{j-1}^2) + t_{j-1}t_j(t_{j+1}-t_{j-1})\right]$$

$$\Gamma\ (t_{j+1}-t_{j-1})^3 = t_{j+1}^3 - 3t_{j+1}^2\, t_{j-1} + 3t_{j+1}\, t_{j-2}^2 - t_{j-1}^3$$

$$\Rightarrow t_{j+1}^3 - t_{j-1}^3 = \underbrace{(t_{j+1}-t_{j-1})^3}_{8\tau^2} + \underbrace{3\, t_{j+1}^2 t_{j-1} - 3t_j\, t_{j-1}^2}_{6 t_{j+1} t_{j-1}}$$

$$\begin{aligned}
u_{j+1} \approx u_{j-1} &+ f_{j-1}\frac{1}{2\tau^2}\Big[\frac{8}{3}\tau^3 + 2t_{j+1}(t_{j-1}+t_j)\tau - \tau\left(t_j t_{j+1} + t_{j+1}^2 + t_j t_{j-1} + t_{j+1} t_{j-1}\right)\\
&- f_j\, \frac{1}{\tau^2}\Big[\frac{8}{3}\tau^3 + 4t_{j+1} t_{j-1}\tau - \tau\left(t_{j-1}^2 + 2t_{j+1} t_{j-1} + t_{j+1}^2\right)\Big]\Big]\\
&+ f_{j+1}\frac{1}{2\tau^2}\Big[\frac{8}{3}\tau^3 + 2t_{j-1}(t_{j+1}+t_j) - \tau\left(t_{j-1}t_{j+1} + t_j t_{j+1} + t_{j-1} t_j + t_{j-1}^2\right)\Big]\Big]\\
= u_{j-1} &+ f_{j-1}\frac{1}{2\tau}\Big[\frac{8}{3}\tau^2 + \left(t_j t_{j+1} - t_{j+1}^2 - t_j t_{j-1} + t_{j+1} t_{j-1}\right)\Big]\\
&- f_j\, \frac{1}{\tau}\Big(\frac{8}{3}\tau^2 - \left(t_{j-1}^2 - 2t_{j+1} t_{j-1} + t_{j+1}^2\right)\Big]\Big]\\
&+ f_{j+1}\frac{1}{2\tau}\Big[\frac{8}{3}\tau^2 + \left(t_{j-1}t_{j+1} - t_j t_{j+1} + t_{j-1} t_j - t_{j-1}^2\right)\Big]\Big]\\
= u_{j-1} &+ f_{j-1}\,\frac{1}{2\tau}\Big[\frac{8}{3}\tau^2 + \overbrace{(t_j - t_{j+1})}^{-\tau}\overbrace{(t_{j+1}-t_{j-1})}^{2\tau}\Big]\\
&+ f_j\,\frac{1}{\tau}\Big[\frac{8}{3}\tau^2 - \underbrace{(t_{j-1}-t_{j+1})^2}_{(-2\tau)^2}\Big]\\
&+ f_{j+1}\frac{1}{2\tau}\Big[\frac{8}{3}\tau^2 + \underbrace{(t_{j-1}-t_j)}_{-\tau}\underbrace{(t_{j+1}-t_{j-1})}_{2\tau}\Big]
\end{aligned}$$

$$= u_{j-1} + \tau\left(\frac{1}{3}f_{j-1} + \frac{4}{3}f_j + \frac{1}{3}f_{j+1}\right)$$