

Numerical Solution of ODEs

Exercise Class

28th November 2023

Predictor/Corrector

For implicit methods we have previously used either a fixed-point or Newton method to solve. Alternatively, we can use the *Predictor/Corrector* method, where we combine a implicit and explicit method:

Predictor An explicit method (e.g. Adams-Bashfort 2)

Corrector An implicit method (e.g. Adams-Moulton 2)

Algorithm (PECE). For the timestep t_{j+1} :

$$\begin{aligned} \text{Predict (ab2):} \quad & u_{j+1}^P = u_j + \tau \left(\frac{3}{2}f(t_j, u_j) - \frac{1}{2}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^P) \\ \text{Correct (am2):} \quad & u_{j+1}^C = u_j + \tau \left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^C) \end{aligned}$$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1}, u_{j+1}) = f_{j+1}^E$.

Algorithm (PEC). For the timestep t_{j+1} :

$$\begin{aligned} \text{Predict (ab2):} \quad & u_{j+1}^P = u_j + \tau \left(\frac{3}{2}f(t_j, u_j) - \frac{1}{2}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^P) \\ \text{Correct (am2):} \quad & u_{j+1}^C = u_j + \tau \left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1}) \right) \end{aligned}$$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1}, u_{j+1}) = f_{j+1}^E$.

Algorithm (PECECE = P(EC)²E). For the timestep t_{j+1} :

$$\begin{aligned} \text{Predict (ab2):} \quad & u_{j+1}^P = u_j + \tau \left(\frac{3}{2}f(t_j, u_j) - \frac{1}{2}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^P) \\ \text{Correct (am2):} \quad & u_{j+1}^C = u_j + \tau \left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^C) \\ \text{Correct (am2):} \quad & u_{j+1}^C = u_j + \tau \left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1}) \right) \\ \text{Evaluate:} \quad & f_{j+1}^E = f(t_{j+1}, u_{j+1}^C) \end{aligned}$$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1}, u_{j+1}) = f_{j+1}^E$.

By repeating the Evaluate/Correct steps k -times, $k \in \mathbb{N}$, we can derive the algorithms $P(EC)^kC$ and $P(EC)^k$.

BDF (Backward Differentiation Formula) Methods

This is an m -step method of the highest-order where $b_0 = \dots = b_{m-1} = 0$ and $a_m = 1$. We derive by solving a linear system:

$$\begin{aligned}
 \text{BDF1 :} & & u_{j+1} - u_j &= \tau f(t_{j+1}, u_{j+1}), \\
 \text{BDF2 :} & & u_{j+2} - \frac{4}{3}u_{j+1} + \frac{1}{3}u_j &= \frac{2}{3}\tau f(t_{j+2}, u_{j+2}), \\
 \text{BDF3 :} & & u_{j+3} - \frac{18}{11}u_{j+2} + \frac{9}{11}u_{j+1} - \frac{2}{11}u_j &= \frac{6}{11}\tau f(t_{j+3}, u_{j+3}), \\
 \text{BDF4 :} & & u_{j+4} - \frac{48}{25}u_{j+3} + \frac{36}{25}u_{j+2} - \frac{16}{25}u_{j+1} + \frac{3}{25}u_j &= \frac{12}{25}\tau f(t_{j+4}, u_{j+4}), \\
 \text{BDF5 :} & & u_{j+5} - \frac{300}{137}u_{j+4} + \frac{300}{137}u_{j+3} - \frac{200}{137}u_{j+2} + \frac{75}{137}u_{j+1} - \frac{12}{137}u_j &= \frac{60}{137}\tau f(t_{j+5}, u_{j+5}), \\
 \text{BDF6 :} & & u_{j+6} - \frac{360}{147}u_{j+5} + \frac{450}{137}u_{j+4} - \frac{400}{147}u_{j+3} + \frac{225}{147}u_{j+2} \\
 & & & - \frac{72}{147}u_{j+1} + \frac{10}{147}u_j &= \frac{60}{147}\tau f(t_{j+6}, u_{j+6}),
 \end{aligned}$$

m -step BDF is order $p = m$. Note that BDF1, ..., BDF6 are D-stable, but for $m \geq 7$ BDF is *not* D-stable.

Exercises

1. Create a method to implement BDF4, and compare the numerical solutions given to BDF2, BDF3, 2-step Adams-Moulton and `ode23` for the following ODEs.

- (a) Linear oscillator (`oscillator.m`) on the time interval $t \in [0, 10]$, with $\tau = 0.1$:

$$\begin{aligned}
 x_1' &= x_2 \\
 x_2' &= -9x + 10 \cos(t), \\
 \mathbf{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}
 \end{aligned}$$

For comparisons plot t vs. x_1 .

- (b) Stiff linear system (`linsystem.m`) on the time interval $t \in [0, 0.05]$ with $\tau = 0.001$:

$$\begin{aligned}
 \mathbf{x}' &= \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \mathbf{x} \\
 \mathbf{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}
 \end{aligned}$$

Also run convergence analysis using `conv_analysis.m` to deduce the order of BDF2, BDF3, and BDF4.

Modify the tolerance `tol` used by the BDF method. How does this effect the convergence rate and numerical solution?

2. Homework 2