Numerical Solution of ODEs

Exercise Class

5th December 2023

Dynamical Systems

Consider Van der Pol's oscillator:

$$x_1' = x_2 \qquad = f_1(x) \tag{1}$$
$$x_2' = -x_1 + 2ax_2 - x_1^2 x_2 \qquad = f_2(x) \tag{2}$$

with initial conditions $x_0 \in \mathbb{R}^2$ at time $t_0 = 0$. In order to solve this we require a *stiff* solver such as `ode23s` or `ode45s`.

Steady State

This problem has a steady state at $\boldsymbol{x}^* = (0,0)^\top \in \mathbb{R}^2$, but is it A-stable? In order to determine this we need to compute the eigenvalues of the Jacobian matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}^*) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}^*) \\ \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}^*) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 2x_1^* x_2^* & 2a - (x_1^*)^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}.$$

From the characteristic polynomial we can compute that the spectrum of A is

$$\sigma(A) = \left\{ a + \sqrt{a^2+1}, a - \sqrt{a^2+1} \right\}.$$

The steady state is A-stable if $\max_{\lambda \in \sigma(A)} \operatorname{Re}(\lambda) < 0$. So we evaluate for different values of a:

$|a| < 1$: Here $\sigma(A) = \{a + bi, a - bi\}$, for some $b \in \mathbb{R}$; therefore, $\max_{\lambda \in \sigma(A)} \operatorname{Re}(\lambda) = a$. Hence, we have that $\boldsymbol{x}^*$ is A-stable if $a < 0$ and $\boldsymbol{x}^*$ is unstable if $a > 0$.

$|a| > 1$: Here $\sigma(A) = \{a + b, a - b\}$, for some $b \in [0, |a|]$; therefore, $\max_{\lambda \in \sigma(A)} \operatorname{Re}(\lambda) = a + b$. Hence, we have that $\boldsymbol{x}^*$ is A-stable if $a < 0$ and $\boldsymbol{x}^*$ is unstable if $a > 0$.

Linearisation

We can generate the linearised version of Van der Pol's oscillator using Taylor's expansion around $\boldsymbol{x}^*$:

$$\boldsymbol{x}' = f(\boldsymbol{x}) = f(\boldsymbol{x}^*) + A(\boldsymbol{x} - \boldsymbol{x}^*) + \underbrace{g(\boldsymbol{x} - \boldsymbol{x}^*)}_{\text{Higher-order terms discarded}}.$$

This gives the linearised form as

$$x_1' = x_2,$$
$$x_2' = -x_1 + 2ax_2.$$

Exercises

1. Consider Van der Pol's oscillator given by (1)–(2).

   (a) Plot the result of solving forwards to T_f and backwards to T_b for the following situations:

      i. $x_0 = (1, 1)^\top$, $a = -0.1$, $T_f = 20$, $T_b = -5$
      ii. $x_0 = (0, 0)^\top$, $a = -0.1$, $T_f = 20$, $T_b = -1.5$
      iii. $x_0 = (1, 1)^\top$, $a = 1.1$, $T_f = 20$, $T_b = -1.5$
      iv. $x_0 = (-1, 6)^\top$, $a = 1.1$, $T_f = 20$, $T_b = -0.4$

   (b) For Van der Pol's oscillator study the steady state numerically (see `vdpol_steady`) for $a = -1.1, 0.5, 1.1$.

2. Consider the initial value problem

$$
\begin{aligned}
x_1' &= (a-b)x_1 - cx_2 + x_1(x_3 + d(1 - x_3^2)) \\
x_2' &= cx_1 + (a-b)x_2 + x_2(x_3 + d(1 - x_3^2)) \\
x_3' &= ax_3 - (x_1^2 + x_2^2 + x_3^2)
\end{aligned}
$$

   with initial conditions $x_0 \in \mathbb{R}^3$ at time $t_0 = 0$. This problem has two steady states,

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x^* = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}.
$$

   (a) Compute the Jacobian $A \in \mathbb{R}^{3\times 3}$ at x^* and check if this steady state is A-stable for:

      i. $a = 1.0$, $b = 3$, $c = 0.25$, $d = 0.2$
      ii. $a = 1.95$, $b = 3$, $c = 0.25$, $d = 0.2$
      iii. $a = 2.02$, $b = 3$, $c = 0.25$, $d = 0.2$

      Remark. *Attempt to deduce the eigenvalues analytically if possible, and then verify by numerically calculating the eigenvalues using the MATLAB* `eig` *function.*

   (b) Plot and estimate the ω-limit for the same parameters

      Hint. *Solve upto time $T = 2000$.*