

Numerical Solution of ODEs

Exercise Class

12th December 2023

Domains of Stability

For the explicit Runge-Kutta methods, we have the following domains of stability:

$$\begin{aligned} s = 1 \text{ (Euler):} & \quad S = \{\mu \in \mathbb{C} : |1 + \mu| < 1\} \\ s = 2 \text{ (Runge):} & \quad S = \left\{ \mu \in \mathbb{C} : \left| 1 + \mu + \frac{1}{2}\mu^2 \right| < 1 \right\} \\ s = 3 : & \quad S = \left\{ \mu \in \mathbb{C} : \left| 1 + \mu + \frac{1}{2}\mu + \frac{1}{6}\mu^3 \right| < 1 \right\} \\ s = 4 : & \quad S = \left\{ \mu \in \mathbb{C} : \left| 1 + \mu + \frac{1}{2}\mu + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4 \right| < 1 \right\} \end{aligned}$$

Exercises

1. Consider the pendulum with dynamical friction:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -k \sin x_1 + \varepsilon x_2 \end{aligned}$$

with initial conditions $x_0 \in \mathbb{R}^2$ at time $t_0 = 0$, setting $k = 1$ and $\varepsilon = -0.2$.

The steady states of this equation are given, for all $n \in \mathbb{Z}$, by

$$\begin{aligned} x_1 &= n\pi, \\ x_2 &= 0. \end{aligned}$$

Calculate which steady states are A-stable (consider odd and even values of n separately).

2. Consider the linearised version of the pendulum with dynamical friction:

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -k & \varepsilon \end{pmatrix}$$

with initial conditions $x_0 \in \mathbb{R}^2$ at time $t_0 = 0$, setting $k = 1$ and $\varepsilon = -0.7$.

Using MATLAB plot the eigenvalues of the matrix A and determine for $\tau = 0.1, 0.2, 0.4, 1$ whether the first, second, third, and fourth order Runge-Kutta methods ($s = 1, 2, 3, 4$) are stable.