Domain of stability

Consider the pendulum with dynamical friction

$$x_1' = x_2$$
$$x_2' = -k\sin x_1 + \varepsilon x_2$$

with initial conditions $x_0 \in \mathbb{R}^2$ at time $t_0 = 0$

Set $k = 1$, $\varepsilon = -0.2$

What are the steady states?

$$x_2 = 0$$
$$-k\sin x_1 + \varepsilon x_2 = 0 \Rightarrow \sin x_1 = 0$$
$$\rightarrow x_1 = n\pi, \; n \in \mathbb{Z}$$

Which of these A-stable? Consider n odd/even seperately.

$$A = \begin{pmatrix} 0 & 1 \\ -k\cos n\pi & \varepsilon \end{pmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda - \varepsilon) + k\cos n\pi$$
$$= \lambda^2 - \lambda\varepsilon + k\cos n\pi$$

$$\Rightarrow \lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4k\cos n\pi}}{2} = -0.1 \pm \sqrt{0.01 - \cos n\pi}$$
$$(\varepsilon = -0.2, \; k = 1)$$

n even ($\cos n\pi = 1$)

$$\lambda = -0.1 \pm \sqrt{0.99}\, i$$

$$\Rightarrow \max_{\lambda \in \sigma(A)} \mathrm{Re}(\lambda) = -0.1 \Rightarrow \text{A-stable}$$

n odd ($\cos n\pi = -1$)

$$\lambda = -0.1 \pm \sqrt{1.01} \Rightarrow \max_{\lambda \in \sigma(A)} \mathrm{Re}(\lambda) = \sqrt{1.01} - 0.1 > 0 \Rightarrow \text{Unstable}$$